Antonine Physics A2

Topic 14C Engineering Physics

How to Use this Book

How to use these pages:

- This book intended to complement the work you do with a teacher, not to replace the teacher.
- Read the book along with your notes.
- If you get stuck, ask your teacher for help.
- The best way to succeed in Physics is to practise the questions.

There are many other resources available to help you to progress:

- Web-based resources, many of which are free.
- Your friends on your course.
- Your teacher.
- Books in the library.

This option, which used to be called Applied Physics, looks at how rotating objects behave under different conditions. This is particularly important to those who are designing rotary machines. You will see how the equations of rotary motion are very similar in form to the equations of linear motion.

It also looks at the Laws of Thermodynamics and how these apply to heat engines. These link to the behaviour of gases that you studied in Topic 13.

This option will be of particular interest to students who want to study Engineering as a degree.

Topic 14C

Option C Engineering Physics

1. Rotary Motion

Tutorial 14C.01 Rotational Dynamics

AQA Syllabus

Contents

14C.011 Introduction to Rotational Dynamics

When we looked at objects moving in a circle in Topic 8, we only considered small objects moving around a central point. The objects themselves were small compared to the radius of their orbit. Examples included model aeroplanes tethered by a string, or planets moving around a star. These objects were held to their paths by **centripetal force** and if that force were stopped, the objects would fly off **tangentially** in a straight line.

These ideas don't work for **rotating** systems in which an object is spinning on its axis. Examples of these include:

- rotors of electric motors.
- flywheels.

You will come across quite a lot of odd looking symbols. Don't worry; they are Greek letters which are used as **Physics codes**. A lot of the equations are identical in concept to the **equations of linear motion** (motion in a straight line) which you came across in Topic 5 (AS).

Let's look at the odd symbols and what they mean:

Symbol	***Pronounced***	***Greek Letter***	***Code for***
α	Alpha	a	acceleration
ω	Omega	long o - ō as in "mōtion"	angular velocity
θ	Theta	"th" as in "therapy"	angle

We will now have a look at some of the terms used in rotational dynamics:

14C.012 Angular Velocity

Angular velocity is an important quantity. Suppose we had a record deck turntable turning at 33 rpm (revolutions per minute). Clearly an object at the rim of the turntable moves with a higher linear speed that an object towards the centre. However both move through the same angle every second. We call this the **angular velocity**, which has the physics code ω. We don't use degrees per second, but **radians per second**. Go back to Further Mechanics Tutorial 1 if you are not sure what a radian is (1 rad » 57°). Radians are **dimensionless** units, so are ignored in unit analysis. Some purists say they should be left out altogether. In these notes I will always include them.

Every revolution, all parts of the object turn through 2π radians.

A common way of expressing the rate of turning is **revolutions per minute**. All the equations we will use need the rate of turning (angular velocity) to be in **radians per second**. Therefore, be a good chap and convert the revolutions per minute to radians per second.

Angular velocity = 2π (rev min^{-1} ÷ 60 s min^{-1}) .... *Equation 1*

Note that rpm is better written min^{-1}. But you won't be penalised for writing 'rpm'.

14C.013 Angular Displacement

Angular displacement is simply the angle turned in any given direction. It is given the code θ and is measured in **radians**.

$$\theta = \omega t$$ .... *Equation 2*

(That was a short sub-section, wasn't it?)

14C.014 Angular Acceleration

If the angular velocity is changing, it is of course accelerating. So, we have a term **angular acceleration**, given the code α. It has the code α, rather than a to distinguish it from linear acceleration.

So, just like (linear) velocity = displacement ÷ time, we can write:

angular velocity = angular displacement ÷ time

$$\omega = \frac{\theta}{t}$$

.... *Equation 3*

And just like acceleration (m s^{-2}) = change in velocity (m s^{-1}) ÷ time interval (s), we can write

angular acceleration (rad s^{-2}) = change in angular velocity (rad s^{-1}) ÷ time interval (s)

$$\alpha = \frac{\Delta\omega}{\Delta t}$$ .... *Equation 4*

All these quantities are **vector** quantities, but the directions are pretty easy, **clockwise** or **anticlockwise**. No horrible sines or cosines. So far pretty easy, what? Now it gets a bit harder...

Remember that there are 2π radians in 1 revolution.

Always make sure that you calculator is set to radians, NOT degree when dealing with rotational quantities.

14C.015 Moment of Inertia

You will know that any mass has a reluctance to move, which we call **inertia**. In linear dynamics, we say that all objects are point masses. We have an analogous situation in rotational dynamics which we call the **moment of inertia**. The moment of inertia is the **measure of the opposition of a rotating body to angular acceleration**. It is given the physics code *I* and its units are kg m^2.

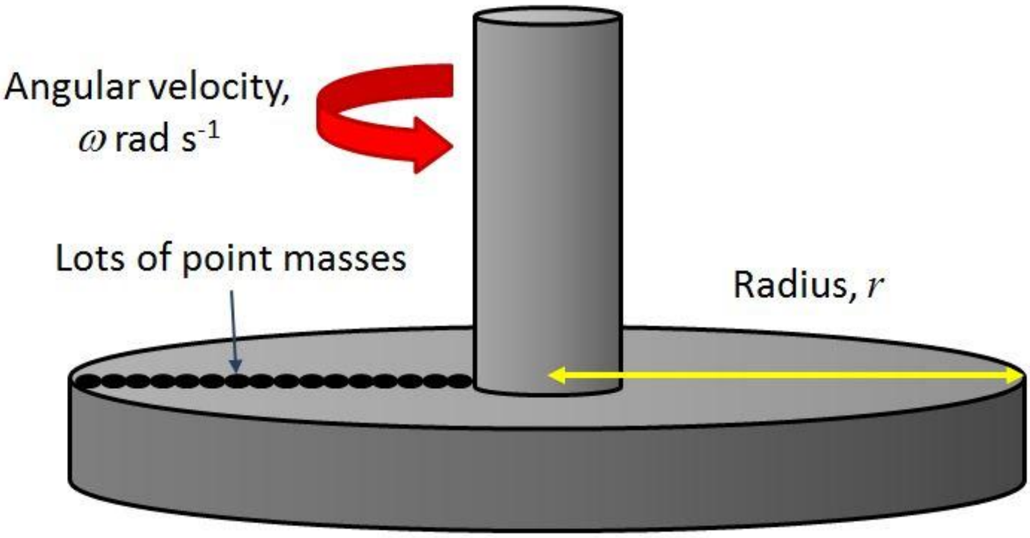


Figure 1 A spinning disc

Suppose we have a disc radius r spinning with angular velocity ω rad s^{-1}. We can think of it as made up of lots of little point masses. We know that the linear speed of each point is v, where

$$v = \omega r$$ .... *Equation 5*

We also know that:

$$E_\mathrm{k} = \frac{1}{2}mv^2$$

.... *Equation 6*

Each little point mass m therefore has a kinetic energy:

$$E_\mathrm{k} = \frac{1}{2}m(\omega r)^2$$

.... *Equation 7*

The total kinetic energy can be found by adding up all the kinetic energies of the little point masses:

$$E_\mathrm{k} = \frac{1}{2}m_1\omega^2{r_1}^2 + \frac{1}{2}m_2\omega^2{r_2}^2 + \frac{1}{2}m_3\omega^2{r_3}^2 \ldots + \frac{1}{2}m_n\omega^2{r_n}^2$$

.... *Equation 8*

We can rewrite this as:

$$E_\mathrm{k} = \frac{1}{2}\omega^2(m_1{r_1}^2 + m_2{r_2}^2 + m_3{r_3}^2 \ldots + m_n{r_n}^2\ )$$

.... *Equation 9*

And we can rewrite this as:

$$E_\mathrm{k} = \frac{1}{2}\omega^2(\Sigma mr^2)$$

.... *Equation 10*

The term Σmr^2 is the sum of all the terms mr^2. The symbol Σ is "Sigma", a Greek capital letter 'S'. This term Σmr^2 is described as the **moment of inertia** and is given the code I.

$$I = \Sigma m r^2$$

.... Equation 11

Its units are **kilogram metre squared** (kg m^2).

For the kinetic energy:

$$E_\mathrm{k} = \frac{1}{2} I \omega^2$$

.... Equation 12

In the exam you are NOT expected to then go on to integrate to derive for particular cases of moments of inertia. The appropriate relationship will be given to you, or the moment of inertia will be given to you already worked out. However, if you want to see this, click to go to Tutorial 14C.07. It is only required for students studying for the Cambridge Pre-U syllabus. The derivations are long and are not intuitive.

For a **circular disc** of mass M and radius r:

$$I = \frac{M r^2}{2}$$

.... Equation 13

For a **solid cylinder**, the relationship is as above.

For a **hollow cylinder open at both ends**, the moment of inertia is:

$$I = M r^2$$

.... Equation 14

For a **solid sphere**, mass *M* and radius *r* the moment of inertia is:

$$I = \frac{2Mr^2}{5}$$

.... *Equation 15*

For a **thin spherical shell**, mass m and radius r the moment of inertia is:

$$I = \frac{2}{3}mr^2$$

.... *Equation 16*

In the past papers I have not yet seen the first or third of these formulae, but I have seen a question that got you to use $I = Mr^2$, ignoring the sigma bit. Other times you have been asked to find I from either the torque or the kinetic energy.

Extension (for SQA Advanced Higher Students only)

For a rod of length l, and mass m, the moment of inertia can be worked out using the relationships below. If you want to see the derivation, see Topic 15.

If the rod is spinning about its centre point like this (*Figure 2*):

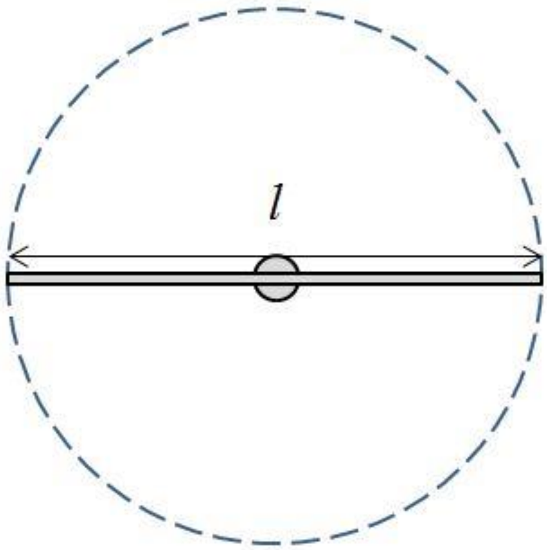


Figure 2 A rod spinning about its centre point

The relationship is:

$$I = \frac{1}{12}ml^2$$

.... *Equation 17*

If the rod is held at one end like this (*Figure 3*):

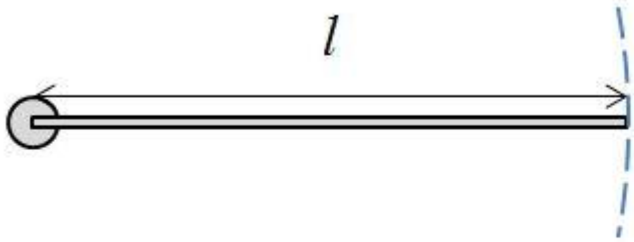


Figure 3 Moment of inertia for a rod held at the end

The relationship is:

$$I = \frac{1}{3}ml^2$$

.... *Equation 18*

The key thing to remember is that the **moment of inertia** is the rotational equivalent of **mass**.

14C.016 Kinetic Energy

The kinetic energy of a spinning object is given by:

$$E_k = \frac{1}{2}I\omega^2$$

.... *Equation 19*

[I - moment of inertia (kg m^2); ω - angular velocity (rad s^{-1})]

Like kinetic energy in linear motion, rotational kinetic energy is a scalar, even though the angular velocity is a vector.

Question 14C.01.5 gets you to consider a problem using this relationship.

To show you that Question 14C.01.5 is not a figment of the examiner's imagination, the picture below (*Figure 4*) shows how a flywheel battery works

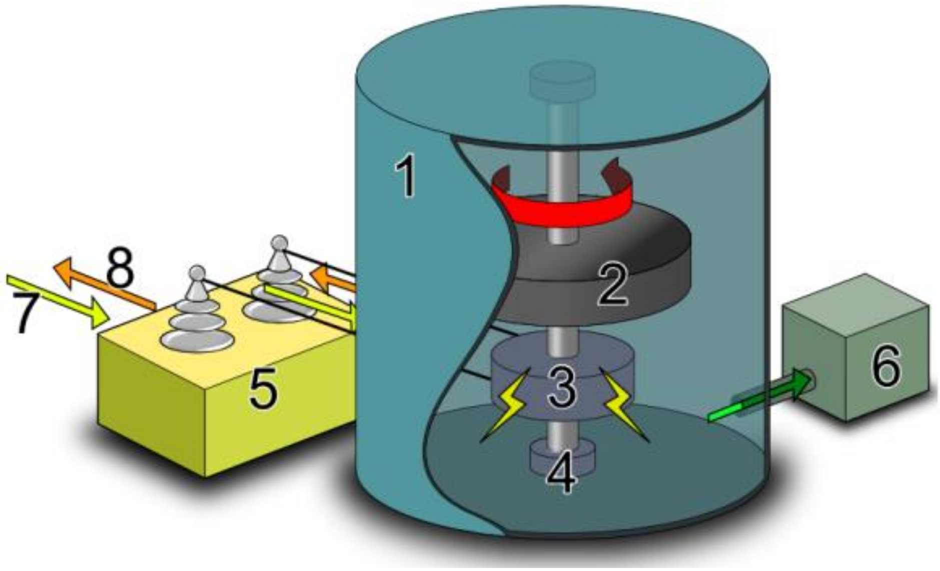


Figure 4 A flywheel battery (Image by Tosaka, Wikimedia Commons)

(1) Case; (2) Flywheel; (3) Motor-generator; (4) Main bearing; (5) Transformer; (6) Vacuum pump; (7) Current in; (8) Current out.

This idea is not new. Third-rail electric locomotives had a motor-generator to help them trundle over gaps in the electric conductor rail.

The **total potential energy** of a rotating system that is in linear motion is the sum of:

- the translational kinetic energy.
- the rotational kinetic energy.

$$\text{Total } E_{\mathrm{p}} = \text{Rotational } E_k + \text{Translational } E_k$$

$$E_p = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 \quad \text{................ } \textit{Equation 20}$$

Worked Example

A solid ball has a mass of 1.50 kg and a diameter of 10.0 cm. It is spinning at 2000 revolutions per minute and is travelling horizontally at a linear speed of 20.0 m s^{-1}.

Calculate:
(a) The moment of inertia.
(b) The rotational kinetic energy.
(c) The total potential energy.

Answer

Radius = 0.050 m. Rate of rotation per second = 33.3 s^{-1}.

Angular velocity = 33.3 × 2π = 209.4 rad s^{-1}.

(a) Formula for the moment of inertia of a sphere:

$$I = \frac{2Mr^2}{5}$$

I = (2 × 1.50 kg × (0.050 m)^2) ÷ 5 = **1.50 × 10^{-3} kg m^2**.

(b) Rotational kinetic energy:

$$E_k = \frac{1}{2}I\omega^2$$

E_k = 1/2 × 1.50 × 10^{-3} kg m^2 × (209.4 rad s^{-1})^2 = **32.9 J**

(c) Translational kinetic energy:

$$E_k = \frac{1}{2}mv^2$$

E_k = 1/2 × 1.50 kg × (20.0 m s^{-1})^2 = 300 J

Potential energy = translational kinetic energy + rotational kinetic energy

= 300 J + 32.9 J = **333 J** (3 s.f)

14C.017 Flywheels

Many machines have a **flywheel** which is a heavy lump of metal spinning on an axis. A single cylinder petrol engine needs a flywheel to keep it running smoothly. If it didn't have one, it would stall on the compression stroke. Four-cylinder car engines would still be jerky without a flywheel. Twelve cylinder car engines have less need for a flywheel, but still have one because:

- It is useful to put the toothed ring for the starter motor onto it.
- It is a useful face for a clutch.

What is the best design for a flywheel? The obvious shape is a solid disk of steel. The diagram shows a couple of cross-sections. Both flywheels have the same mass and mean radius. Both have a small **flange** of very small mass so that they can be bolted to the shaft.

- A solid disk flywheel (*Figure 5*):

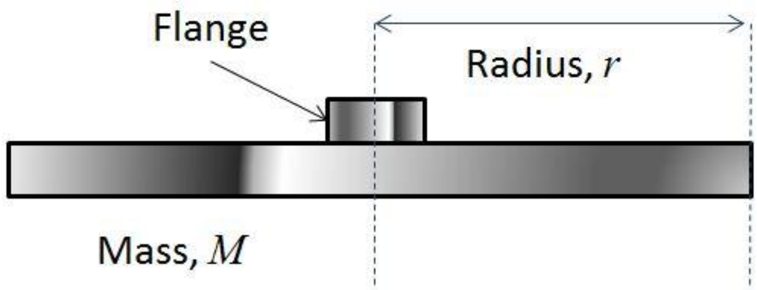


Figure 5 A solid disc flywheel

- A flywheel that has a thin plate in the middle and most of its mass as a ring around the outside (*Figure 6*):

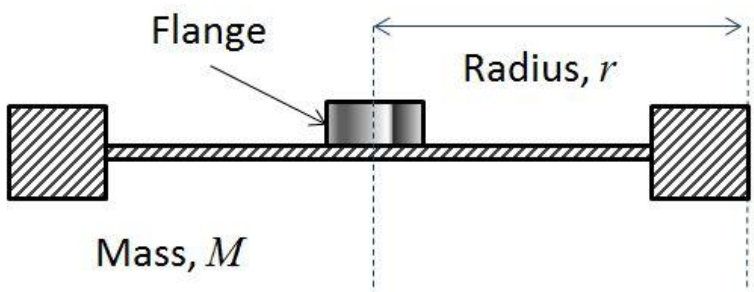


Figure 6 A ring flywheel

The flywheel battery in 14C.01.5 would have a flywheel of the second design.

14C.018 Equations of Rotational Motion

There are four of these, each of which has an equivalent of linear motion, and are used in exactly the same way. The only difference is the code for some of the terms, and if you are at ease with angular velocity, angular displacement, and angular acceleration, you will have no difficulty at all with these. If you have forgotten about the linear equations of motion, go back to Topic 5 and revise them.

This table shows the terms and what they mean:

Term	*What it means*	*Units*
ω_1	Angular velocity at start	rad s^{-1}
ω_2	Angular velocity at end	rad s^{-1}
θ	Angular displacement	rad
α	Angular acceleration	rad s^{-2}
t	time	s

(1) Used to link a second angular velocity to the first, acceleration and time.

$$\omega_2 = \omega_1 + \alpha t$$

.... *Equation 21*

It is used just like $v = u + at$

(2) Used to link the angular displacement to the angular velocity at the start, with the time, and the angular acceleration.

$$\theta = \omega_1 t + \frac{1}{2}\alpha t^2$$

.... *Equation 22*

It is used just like $s = ut + 1/2\ at^2$.

(3) Used to link the angular velocity at end with the first angular velocity, the angular displacement and the acceleration.

$$\omega_2{}^2 = \omega_1{}^2 + 2\alpha\theta$$

.... *Equation 23*

It is used just like $v^2 = u^2 + 2as$.

(4) This equation is used to link the angular displacement to the average of two angular velocities and the time interval:

$$\theta = \frac{(\omega_1 + \omega_2)t}{2}$$

............*Equation 24*

It is used just like $s = (u + v)t/2$

14C.019 Torque and Angular Acceleration

You will be familiar with $F = ma$. Rotational motion has an identical relationship. You will remember that a **torque** is a turning force (*Figure 7*).

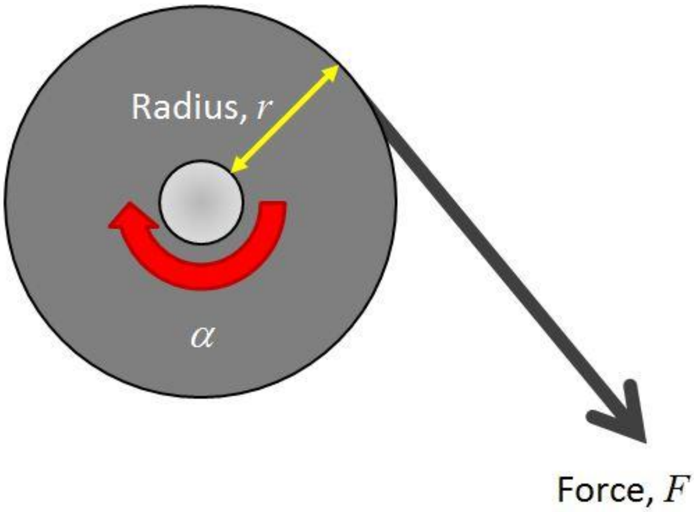


Figure 7 Torque, a turning force

The diagram shows a cord wrapped around a pulley (like the pull-cord used to start a lawn mower). The pulling force F causes a **torque** τ (tau, a Greek lower case letter 't', the physics code for torque) which can easily be worked out by:

$$\tau = Fr$$ .... *Equation 25*

If we double the pull, we will double the torque which means that the angular acceleration will be doubled. However, if we look at the mass, we find that it's not the only factor. The shape of the pulley is important, so instead of mass, we use the **moment of inertia**.

So, if we apply a torque to a rotating body, it will undergo angular acceleration. In other words, it will spin faster. The angular acceleration due to a torque is given by:

$$\tau = I\alpha$$ .... *Equation 26*

[τ - torque (N m); I - moment of inertia (kg m^2); α - angular acceleration (rad s^{-2})]

Let's sum up by comparing **linear motion** with **rotational motion**:

Linear Motion			***Rotational Motion***		
Quantity	***Code***	***Unit***	***Quantity***	***Code***	***Unit***
Displacement	s	m	Angular displacement	θ	rad
Velocity	v	m s^{-1}	Angular velocity	ω	rad s^{-1}
Acceleration	a	m s^{-2}	Angular Acceleration	α	rad s^{-2}
Mass	m	kg	Moment of Inertia	I	kg m^2
Force	F	N	Torque	τ	N m

We will compare equations:

Linear Motion	Rotational Motion
$v = \frac{\Delta s}{\Delta t}$	$\omega = \frac{\Delta \theta}{\Delta t}$
$a = \frac{\Delta v}{\Delta t}$	$\alpha = \frac{\Delta \omega}{\Delta t}$.
$F = ma$	$\tau = I\alpha$
$v = u + at$	$\omega_2 = \omega_1 + \alpha t$
$s = ut + \frac{1}{2}at^2$	$\theta = \omega_1 t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2as$	${\omega_2}^2 = {\omega_1}^2 + 2\alpha\theta$
$s = \frac{(u+v)t}{2}$	$\theta = \frac{(\omega_1 + \omega_2)t}{2}$

You select for the equation of rotational motion in exactly the same way as you would for an equation for linear motion. Suppose you had a problem where you had time, the angular velocity at the start and end, and you were asked to find the angular acceleration. You would choose (and rearrange):

$$\omega_2 = \omega_1 + \alpha t$$

14C.0110 Using Calculus in Rotational Dynamics (SQA AH and Extension)

This subsection is for students of the SQA Advanced Higher syllabus only. For students of other A-level syllabi, you are NOT expected to know this. However, you may want to study this as an extension. These notes may be helpful if you are intending to study Engineering as a university level course. Do not bring it up in the exam, in case you get it wrong. You have been warned!

In linear dynamics (Topic 8) we saw how we can use calculus to establish relationships between quantities like acceleration, velocity, and displacement. We can do the same in rotational dynamics. We do not get sudden **transitions** from angular acceleration to constant angular velocity, with sudden graphical **inflections**. At university level, a **calculus** treatment is preferred. Calculus is a mathematical technique that allows us to work out the gradient of a graph or the area under the graph, if we know the relationship between two quantities.

In rotational dynamics you will see the following:

1. Angular velocity, ω, is the rate of change of angular displacement θ. In **calculus notation**, this is written as:

$$\omega = \frac{d\theta}{dt}$$

.... *Equation 27*

2. Angular acceleration, α, is the rate of change of angular velocity, ω. In calculus notation, this is written as:

$$\alpha = \frac{d\omega}{dt}$$

.... *Equation 28*

So, what is the difference? If we are talking about a constant rate of change, there is no difference. However, if the rate of change is variable, the calculus notation is used for an **instantaneous** change. This is summed up in the graph in *Figure 8*.

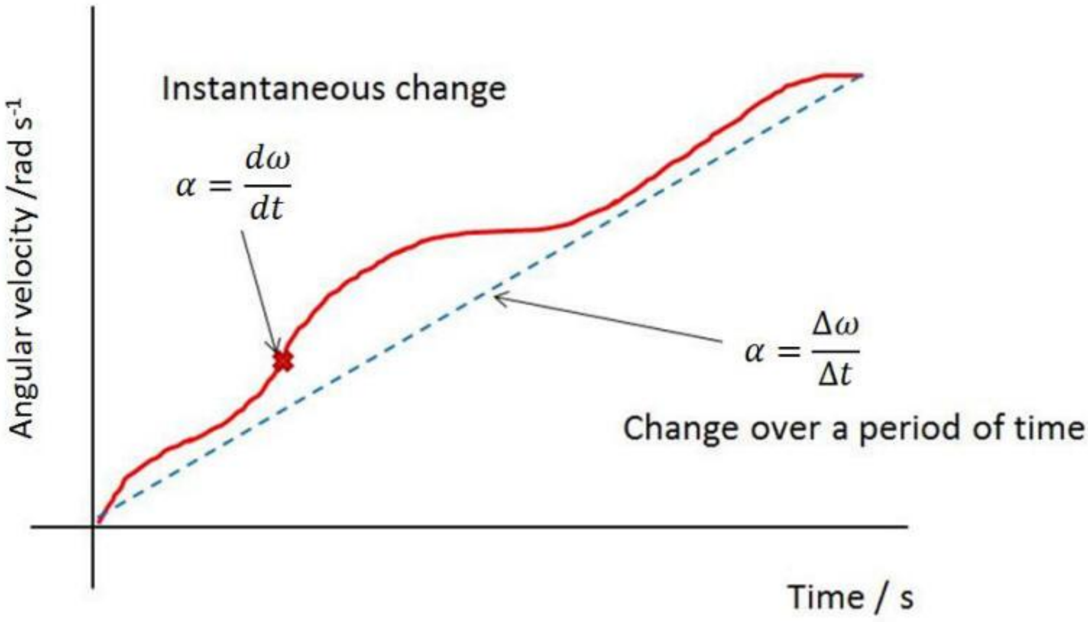


Figure 8 Showing the difference between constant change and instantaneous change

In this graph we see a very irregular increase in the angular velocity of an object. We can look at the rate of change in velocity at a particular instant, or we look at the overall change in velocity over a longer period. The instantaneous change in the angular velocity is represented by the term $d\omega$, while the overall change is represented by $\Delta\omega$.

We could, of course, take a tangent from the graph at the particular instant and measure the gradient of the tangent. However, it is likely that there will be uncertainty. With differentiation, there is no uncertainty.

Acceleration is related to displacement using:

$$\alpha = \frac{d^2\theta}{dt^2}$$

.... *Equation 29*

This is a **second derivative**, which means a derivative of a derivative.

In summary:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

.... *Equation 30*

Using Differentiation

Differentiation is about finding the **gradient** of the graph. As we have seen above:

- Velocity is the gradient of the displacement time graph.
- Acceleration is the gradient of the velocity time graph

Maths Note

There are two things that we can do with Calculus:

1. We can **differentiate**, which means that we find the gradient of the graph of a known relationship.

2. We can **integrate**, which means that we find the area under the graph of a known relationship.

We do these mathematically without having to draw the graph.

This is NOT a comprehensive treatment of calculus, but I hope it will help you how to use it in kinematics calculations

Differentiation

Differentiation is about determining the gradient of a graph.

There are a number of **rules** of differentiation. We will use only two here.

- Added constants differentiate to 0.
- Powers differentiate according to this formula:

$$\frac{d}{dx}(x^n) = n(x^{n-1})$$

- Multiplied constants are multiplied with the result of the formula that has been differentiated. Suppose the constant is b:

$$\frac{d}{dx}(bx^n) = bn(x^{n-1})$$

Let us suppose we have a straight line graph that follows the general relationship:

$$y = mx + c$$

If we want to differentiate this, we get:

$$\frac{dy}{dx}(mx + c) = (m \times 1 \times x^0) + 0$$

Therefore:

$$\frac{dy}{dx}(mx + c) = m$$

This tells us that the gradient is m.

We can use calculus to work out the velocity at a particular instant. Here is a graph showing the displacement of an object subject to constant acceleration. This graph has been drawn using the equation:

$$\theta = \frac{1}{2}\alpha t^2$$

.... *Equation 31*

The value of the angular acceleration is 4.0 rad s^{-2} (*Figure 9*).

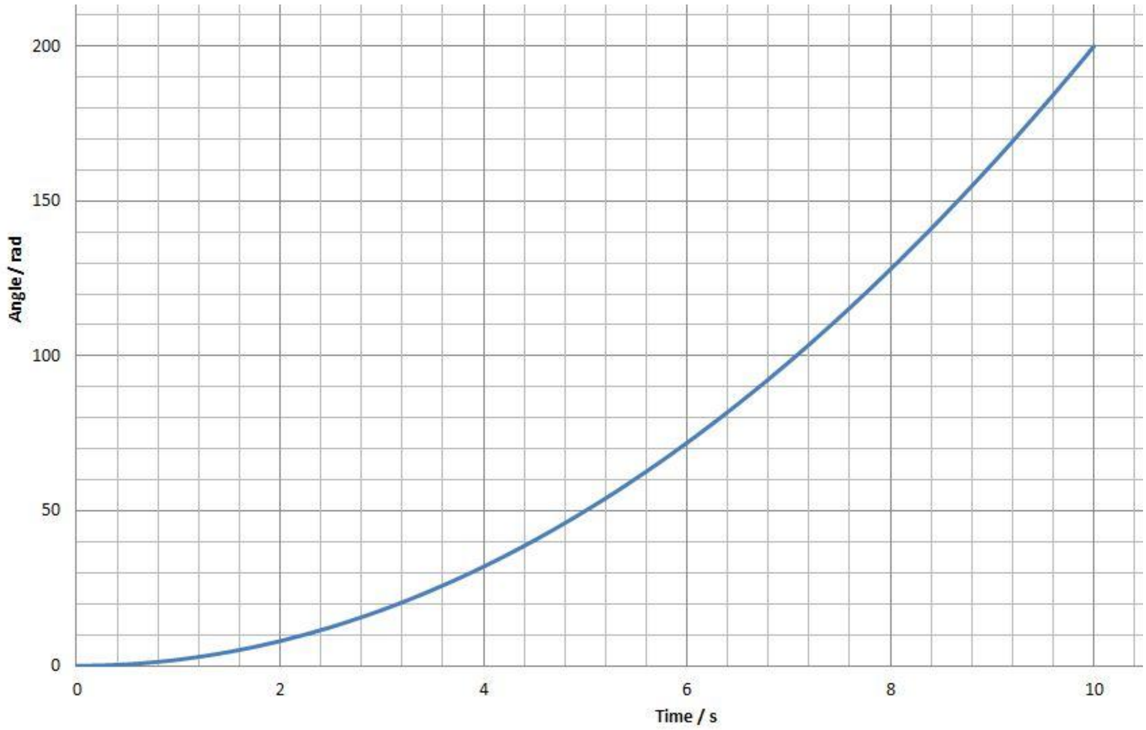


Figure 9 Graph of angular displacement against time

We could, of course, work the gradient of the tangent to give us the velocity at exactly 6.0 s, but there will be uncertainty. Instead, let's use calculus:

$$\omega = \frac{d\theta}{dt} = 2 \times \frac{1}{2}\alpha t = \alpha t$$

.... *Equation 32*

Therefore, we can substitute:

ω = 4.0 rad s^{-2} × 6.0 s = **<u>24 rad s^{-1}</u>**

This is a lot easier that drawing a tangent, then measuring the rise and the run.

We can differentiate the equation:

$$\theta = \omega_1 t + \frac{1}{2}\alpha t^2$$

.... *Equation 33*

As below:

$$\omega = \frac{d\theta}{dt} = 1 \times \omega_1 t^0 + 2 \times \frac{1}{2}\alpha t = \alpha t$$

.............*Equation 34*

To give us:

$$\omega_2 = \omega_1 + \alpha t$$

.... *Equation 35*

Using Integration

Integration is about finding the **area** under the graph. In Topic 8 we have seen how the kinematics equations have been worked out using the area under the velocity time graph. The same applies to rotational dynamics. For some questions we have simply counted the squares under the graph. The counting of squares is both tedious and prone to uncertainty. In simple graphical treatments, acceleration is counted as constant. In reality it is not. Therefore, a mathematical approach is more satisfactory, as it can be quicker and is less prone to uncertainty.

Consider a rotor of an electric car motor accelerating at a rate of α rad s^{-2} over a period of t s. This is a real world situation, and the acceleration is not constant, but reduces because of the increase in friction from the road and the air resistance. See *Figure 10*.

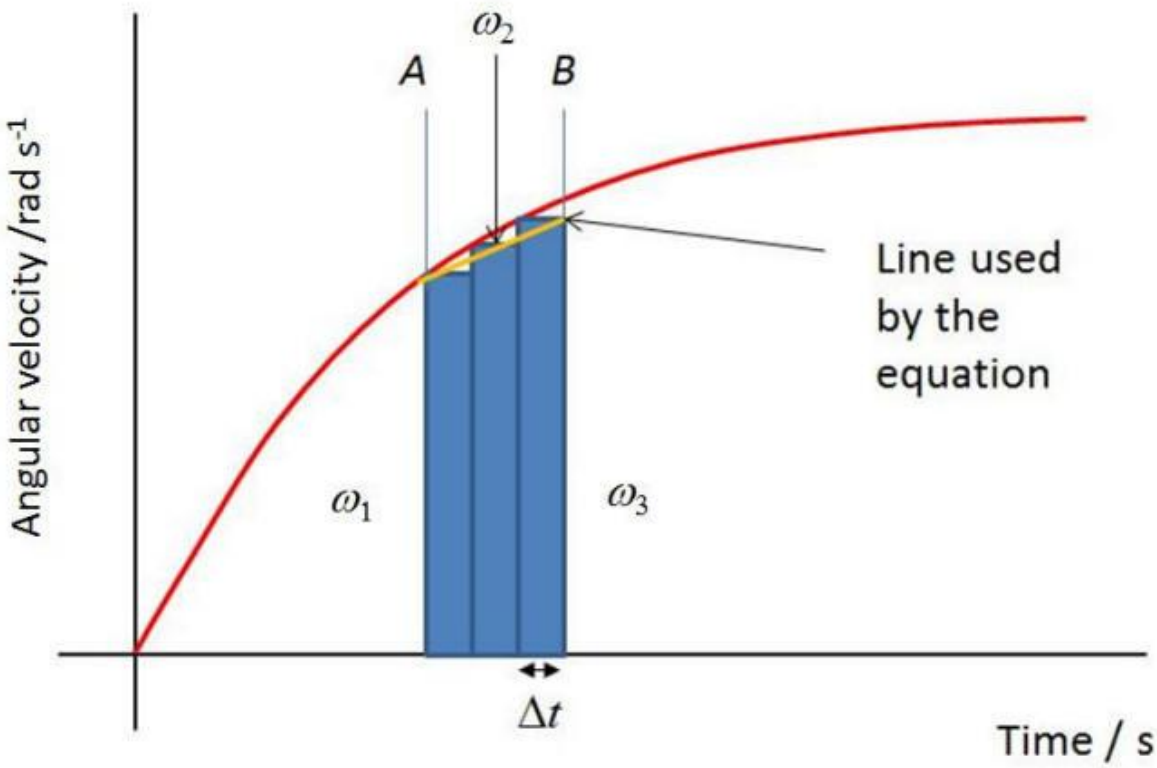


Figure 10 Graph of angular velocity against time

Suppose we want to find the angular displacement θ between the points *A* and *B*. We know that angle turned is the area under the graph. We could use the equation:

$$\theta = \frac{(\omega_1 + \omega_3) \times 3\Delta t}{2}$$

.... *Equation 36*

The equation works out the area under the orange line in the graph. The answer it would give would be too low.

Alternatively, we could break the area into three little strips as shown in the graph. So, we could work out the area of each little strip and add them together:

$$\theta = \omega_1 \Delta t + \omega_2 \Delta t + \omega_3 \Delta t$$

.... *Equation 37*

This will give us a better answer, but it will only be an approximation. We could make the strips narrower, using a shorter time interval (*Figure 11*).

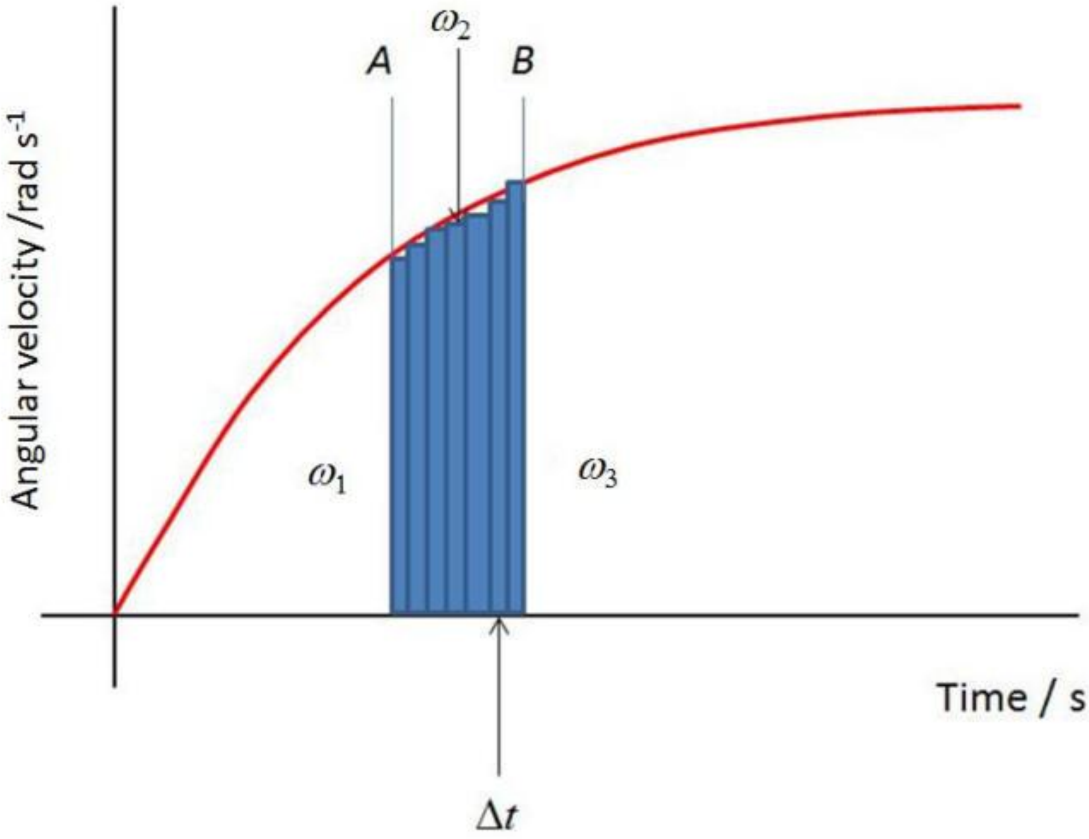


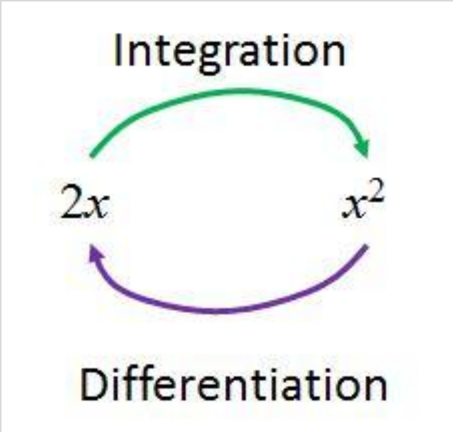
Figure 11 Refining the finding of area under the graph

Therefore, the answer gets closer to the real answer, but it is still an approximation. However, if we make the time interval infinitesimally small, we end up with the true answer. This we do by the process of integration. Instead of writing the width of each strip a D*t*, we write *dt*. Integration adds up all the little strips to give us the true answer.

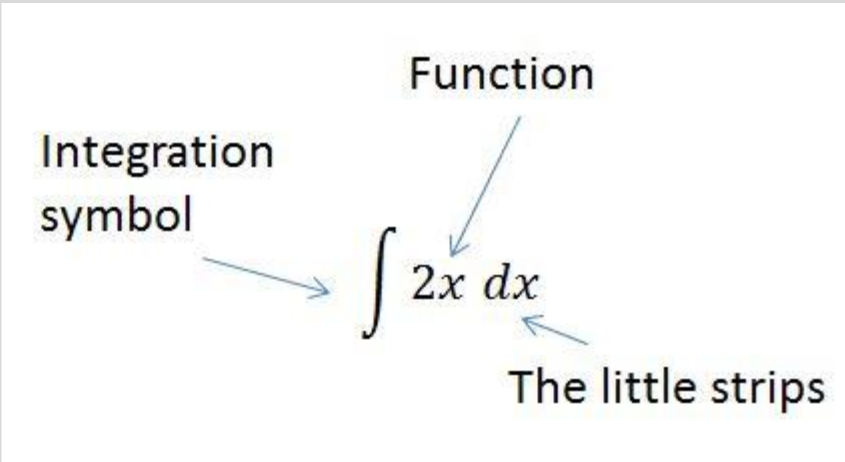
Maths Note

Integration

Integration is the reverse process of differentiation. The idea is shown in the picture below:



The $2x$ term is the **function**. The function to be integrated is sometimes called the **integrand**. Therefore, we write this in calculus form as:



The dx term shows that the little strips go along the x-axis. The integration symbol is a fancy capital letter 'S', which means "summed together". So, we now write:

$$\int 2x\,dx = x^2 + C$$

The C term is a **constant**. When we differentiated, the constant that was added to the function had a differentiated value of 0. Now we are applying the process in reverse, we need to have a definite value for the constant.

Here are some rules for integration:

- A constant is added to the integrated function. In some cases, this might be zero. In other cases, it has a definite value.
- Constants that multiply a function are multiplied with the result of the integrated function. Suppose we have a constant, b:

$$\int 2bx\ dx = bx^2 + C$$

- The power rule is *shown* below. It does not work with x^{-1}.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

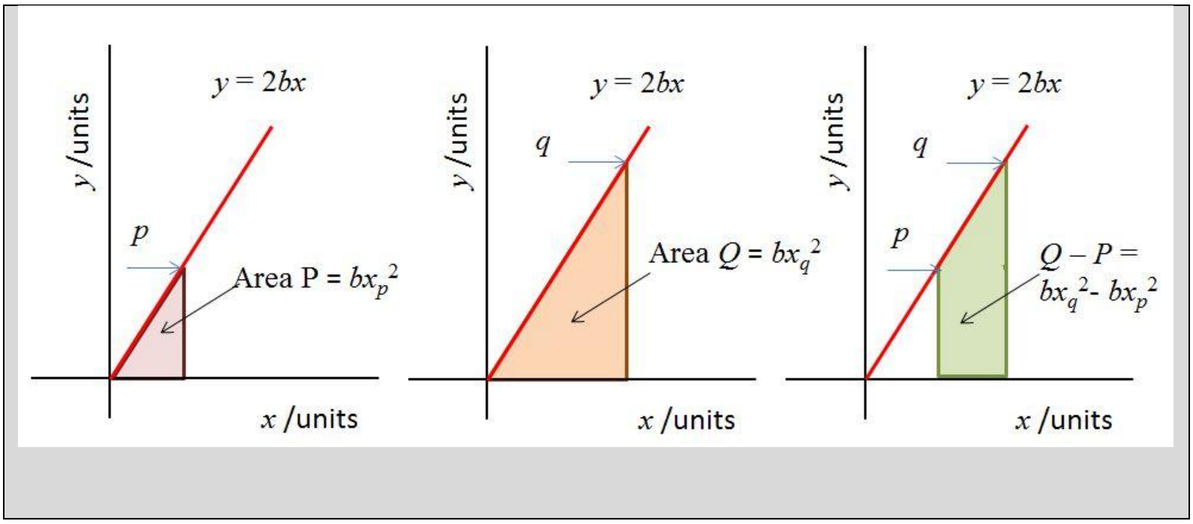
- The integral of x^{-1} is shown below:

$$\int x^{-1} dx = \ln(x) + C$$

Often you will need to integrate between two points. You may see an equation like this:

$$\int_p^q 2bx\ dx = bx^2 + C$$

This means you have to work out the value of the integral at p and the value of the integral at q and then subtract one from the other. This is shown in the picture below. The constant, $C = 0$ for the sake of this argument.



Consider an object moving with a constant angular acceleration, α rad s^{-2} from an initial angular velocity at $t = 0$, ω_1 rad s^{-1}, to a final angular velocity, ω_2 rad s^{-1} at time t. This is shown on the graph below (*Figure 12*):

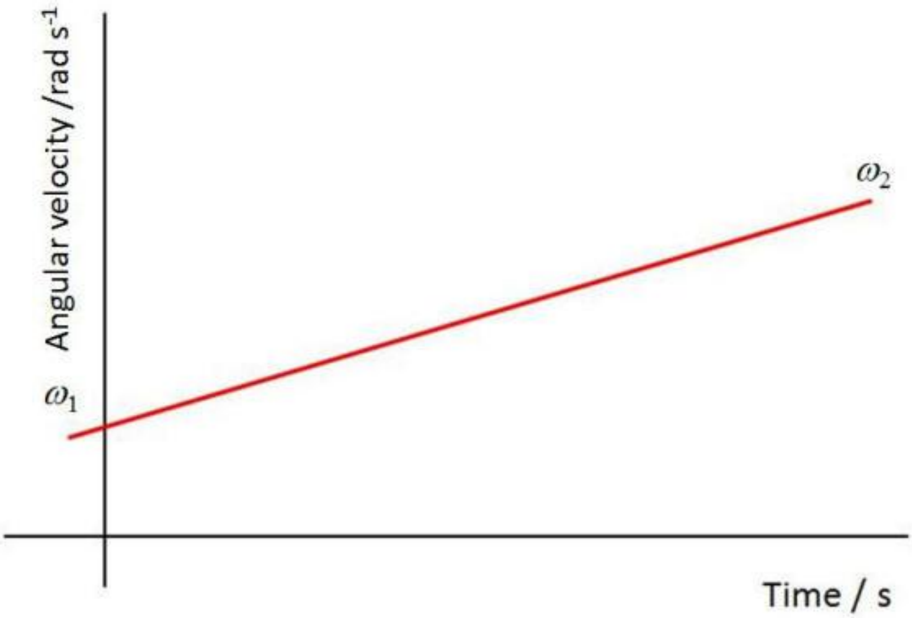


Figure 12 Graph of constant change in angular velocity against time

We know that this graph shows the equation:

$$\omega_2 = \omega_1 + \alpha t$$

.... *Equation 38*

We also know that the area under the graph is the displacement. So, we can integrate, since we know that:

$$\omega = \frac{d\theta}{dt}$$

.... *Equation 39*

So, we can write:

$$\theta = \int_0^{\infty} \omega + \alpha t \, dt = \frac{\omega t^{0+1}}{0+1} + \frac{\alpha t^{1+1}}{1+1} = \omega t + \frac{\alpha t^2}{2}$$

........*Equation 40*

Therefore:

$$\theta = \omega_1 t + \frac{1}{2}\alpha t^2$$

.............*Equation 41*

So, let's use that result:

Worked Example An object has an initial angular velocity of 6.5 rad s^{-1}. At time t = 0 it accelerates at a rate of change of angular velocity of 2.85 rad s^{-1}. What is the angle turned between time t = 3.6 s and time t = 8.5 s?
Answer We can write this in calculus notation: $$s = \int_{3.6\text{ s}}^{8.5\text{ s}} 6.5\text{ m s}^{-1} + 2.85\text{ m s}^{-2} t \, dt$$ Therefore: $$\theta = \left[(6.5\text{ rad s}^{-1} \times 8.5\text{ s}) + \left(\frac{1}{2} \times 2.85\text{ rad s}^{-2} \times (8.5\text{ s})^2\right)\right] - \left[(6.5\text{ rad s}^{-1} \times 3.6\text{ s}) + \left(\frac{1}{2} \times 2.85\text{ rad s}^{-2} \times (3.6\text{ s})^2\right)\right.$$ $$\theta = 158.2\text{ rad} - 41.87\text{ rad} = 116\text{ rad}$$

Questions

Tutorial 14C.01

14C.01.1

What is the angular velocity of a hi-fi record deck playing at 33 rpm?

14C.01.2

What do you think the units are for angular acceleration?

14C.01.3

It takes a motor 5 seconds to accelerate from rest to 3000 rpm. What is its angular acceleration?

14C.01.4

A circular disc and a solid sphere each has a mass of 2.5 kg and a radius of 0.2 m. What is the moment of inertia for each one?

14C.01.5

A flywheel battery can be used in place of lead-acid batteries to provide a short-term electrical power supply when mains power fails. Energy is stored as rotational kinetic energy in a rapidly spinning rotor, which is driven up to speed by a mains-powered motor. To recover the energy, the motor is operated in reverse as a generator driven by the spinning rotor.

The rotor of a flywheel battery is a thin-walled circular cylinder of mass 160 kg and mean radius 0.34 m. It can be rotated at a maximum safe angular speed of 44 000 rev min^{-1}.

Calculate:

(a)

the moment of inertia of the rotor about its own axis,

(b)

the rotational kinetic energy stored in the rotating rotor when it spins at 44 000 rev min^{-1}.

(AQA past question)

14C.01.6

By considering the amount of kinetic energy each one can store, which one is the better design? State what assumptions you make.

14C.01.7

A car wheel is being tested for balance. It is spinning at 750 rpm and the machine then accelerates it to 1500 rpm over a period of 3 s.

What is the angular acceleration?

14C.01.8

Use your result from Question 14C.01.7 to work out the angular displacement through which the wheel turns as it accelerates from 750 rpm to 1500 rpm.

14C.01.9

In an experiment to test a "crash-proof" fuel, an aeroplane was deliberately crash-landed onto spikes which were designed to slice the fuel tanks open. The test was a failure because the fuel burst into flames. The reason for this was because one of the engines was stopped by a spike. The turbine was running at 30 000 rpm and was stopped within 1/3 revolution. The kinetic energy of the engine was dissipated as intense heat which acted as an ignition source for the fuel.

What was the angular deceleration suffered by that engine?

14C.01.10

A car wheel is being tested for balance. It is spinning at 750 rpm and the machine then accelerates it to 1500 rpm over a period of 3 s. Use the equation:

$$\theta = \frac{(\omega_1 + \omega_2)t}{2}$$

to work out the angular displacement. How does your answer compare to the answer you worked out in Question 14C.01.8?

14C.01.11

The wheel of a large dumper truck has a moment of inertia of 10 000 kg m^2 and is being tested by being rotated at 60 rpm. It is brought to rest in a time of 40 s.

(a) What is the initial angular velocity in rad s^{-1}?

(b) What is the angular acceleration?

(c) What is the angular displacement in the first 20 s?

(d) What is the applied torque?

Tutorial 14C.02 Angular Momentum and Power	
AQA Syllabus	
Contents	
14C.021 Angular Momentum	14C.022 Angular Impulse
14C.023 Work and Power in Rotating Objects	

14C.021 Angular Momentum

In Topic 5 we met **linear momentum** as the **product of mass and velocity**, the simple equation being $p = mv$. Any object moving in a straight line has a momentum, so it is very reasonable that any spinning object has angular momentum in its spin. As we saw in 14C.01, we found that the quantity in rotational dynamics that was equivalent to **mass** was the **moment of inertia**. So, we can write an expression that tell us about angular momentum:

Angular momentum = moment of inertia × angular velocity

In code:

$$L = I\omega$$ .... *Equation 42*

[L - angular momentum (kg m^2 s^{-1}); I - moment of inertia (kg m^2); ω - angular velocity (rad s^{-1})]

Cricketers add spin to the ball when they bowl by giving the ball a quick flick just as the ball leaves their hand. The idea is to make it change the direction of flight as it bounces. (I was hopeless at cricket and never got the knack… For this and many other reasons I developed a heart-felt loathing for cricket, which I have to this day.)

Like momentum in a straight line, **angular momentum is conserved**, which means that momentum before = momentum after, provided that there has been no angular impulse. You can try this for yourself on a swivel chair. Spin yourself around. Put your legs out and you will find yourself spinning more slowly. Then tuck your legs in, and you will spin faster. You are changing the moment of inertia. If the moment of inertia increases, the angular velocity must go down **to keep the angular momentum the same.**

High divers and ice skaters use the ideas of angular momentum as part of their acts, although not that many know about the physics.

Worked Example A ballet dancer spins about her vertical axis at 1 revolution per second with her arms outstretched. With her arms folded, her moment of inertia decreases to 40 % of what it was. What is her new rate of turning?
<u>Answer</u> Let her original moment of inertia be I, so her new moment of inertia = 0.4 I Angular momentum is conserved. $I \times 1\ \text{s}^{-1} = L = 0.4\,I \times \omega$ $\omega = 1 \div 0.4 =$ **<u>2.5 revolutions per second</u>**. Note that the turning rate has been kept at revolutions per second. This is alright, but you must be consistent. (Strictly speaking, the answer = 5π rad s^{-1} = 15.7 rad s^{-1}.)

<u>14C.022 Angular Impulse</u>

In Mechanics we saw that **impulse** was **change in momentum**. We could rewrite Newton II as:

$$F = \frac{\Delta p}{\Delta t}$$

.... *Equation 43*

Therefore, impulse is given by rearranging *Equation 43*:

$$\Delta p = F\Delta t$$ .... *Equation 44*

We can do exactly the same with angular momentum. We call the change in angular momentum the angular impulse. Instead of Force, we have torque. Therefore:

Angular impulse (kg m^2 rad s^{-1}) = torque applied (N m) × the time interval (s)

In physics code we write it as:

$$\Delta L = \tau \Delta t$$

.............*Equation 45*

[ΔL = angular impulse (kg m^2 rad s^{-1}); τ = torque (N m); t = time interval (s)]

A small torque applied for a long time interval will have the same effect as a large torque applied for a short time interval.

Here is a rotating sign that rotates in windy conditions. You often see them outside garages (*Figure 13*).

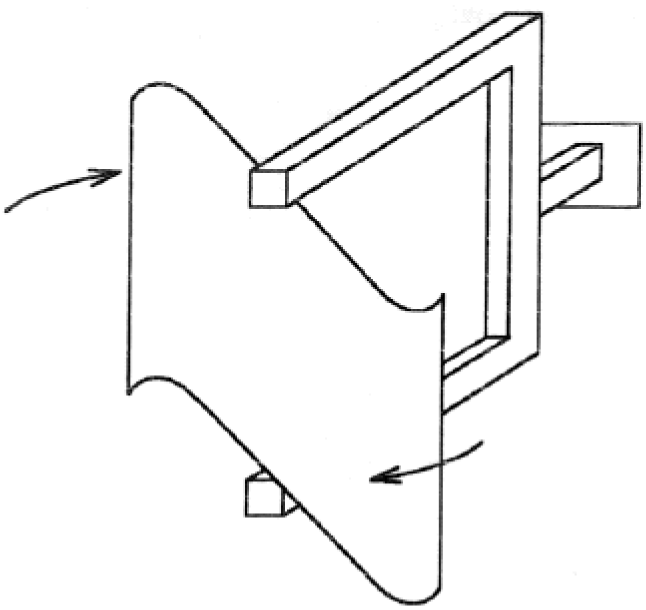


Figure 13 A rotating sign

14C.023 Work and Power in Rotating Objects

In linear motion we have seen that work done can be related by the simple equation:

Work done (J) = Force (N) × distance moved in the direction of a force (m)

$$W = Fs$$ .... *Equation 46*

We can use the same idea to work out the work done in a rotating system:

Work done (J) = torque (N m) × angle rotated (rad)

$$W = \tau\theta$$ .... *Equation 47*

In linear motion we also found a useful relationship linking force and power:

Power (W) = force (N) × speed (m s^{-1}).

We can derive a similar expression for rotational motion:

- Work done = energy used.
- Power = energy used ÷ time interval
- Power = (torque x angle rotated) ÷ time interval

$$P = \frac{\tau\Delta\theta}{\Delta t}$$

.... *Equation 48*

But:

$$\omega = \frac{\Delta\theta}{\Delta t}$$

.... *Equation 49*

So, it does not take a genius to see:

Power (W) = torque (N m) × angular velocity (rad s^{-1})

$$P = \tau\omega$$ .... *Equation 50*

All rotating machines have a certain amount of **friction**. We treat friction as an opposing rotational couple which provides a torque in the opposite direction. We use the equation:

$$\tau = I\alpha$$ .... *Equation 51*

to work out the rate of change of angular velocity.

Worked Example A roller is running freely on its bearings and is turning at a constant angular velocity of 8.5 rad s^{-1}. The moment of inertia of the roller is 2.5 kg m^2. If the frictional couple is 0.67 N m, (a) what is the angular acceleration? (b) How many revolutions does the roller need to come to rest?
Answer (a) Use the relationship $\tau = I\alpha$ The frictional couple should be negative, as it is opposing the turning. -0.67 N m = 2.5 kg m^2 α α = -**0.268 rad s^{-2}** (minus because the roller is slowing down) (b) To work out the angle covered, we use: $\omega_2^2 = \omega_1^2 + 2\alpha\theta$ 0 = 8.5 rad s^{-1} + 2 × -0.268 rad s^{-2} × θ θ = **15.9 rad** No of revolutions = 15.9 rad ÷ 2π rad = **2.5 revolutions**

Questions

Tutorial 14C.02

14C.02.1

Why are the units for L kg m^2 s^{-1}?

14C.02.2

A disc of moment of inertia 10 kg m^2 rotates at an angular velocity of 20 rad s^{-1}. What is its angular momentum?

14C.02.3

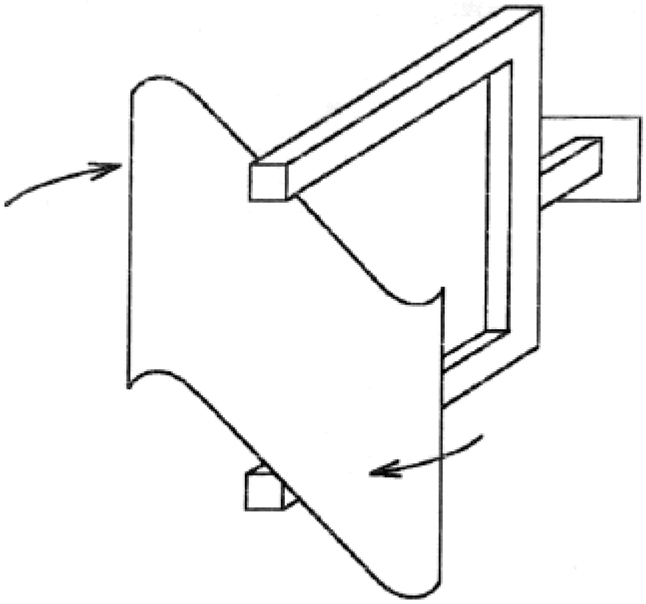
A potter in an African village makes large clay pots on a stone wheel. The wheel rotates freely on a central bearing and is driven by the potter, who applies a tangential force repeatedly to its rim using his foot until the wheel reaches its normal working angular speed. He then stops driving and throws a lump of clay onto the centre of the wheel.

The normal working angular speed of the wheel is 5.0 rad s^{-1}. The moments of inertia of the wheel and the clay about the axis of rotation are 1.6 kg m^2 and 0.25 kg m^2, respectively. When the clay is added, the angular speed of the wheel changes suddenly. The net angular impulse is zero. Calculate the angular speed of the wheel immediately after the clay has been added.

(AQA past question)

14C.02.4

Consider this rotating sign.



On a still day, a gust of wind from a passing vehicle imparts an angular impulse of 1.2 kg m^2 rad s^{-1} to the sign, which accelerates from rest during a time of 2.8 s. The moment of inertia of the sign about its axis of rotation is 4.8×10^{-2} kg m^2.

Assuming that the frictional couple acting on the sign is negligible, calculate:

(a) the angular momentum acquired by the sign as a result of the angular impulse, showing your reasoning clearly,

(b) the angular speed of the sign immediately after the impulse has been imparted.

(c) What was the torque that acted on the sign?

(AQA Past Question)

14C.02.5

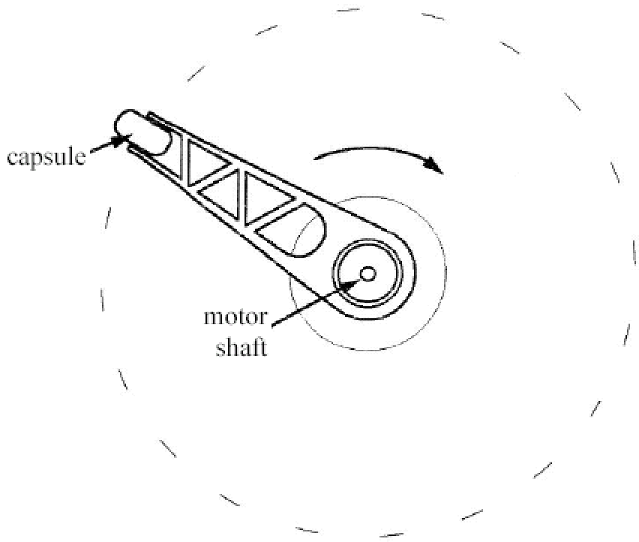
A torque of 135 N m is required to turn a nut through half a turn. What is the work done by the mechanic?

14C.02.6

A motor gives out a torque of 150 Nm at a speed of 3000 rpm. What is its power?

14C.02.7

The diagram shows a human centrifuge used in pilot training to simulate the large "g" forces experienced by pilots during aerial manoeuvres. The trainee sits in the capsule at the end of the rotating centrifuge arm, which is driven by an electric motor.



When working at maximum power, the motor is capable of increasing the angular speed of the arm from its minimum working speed of 1.6 rad s^{-1} to its maximum speed of 7.4 rad s^{-1} in 4.4 s.

The net power needed to achieve this acceleration is 150kW.

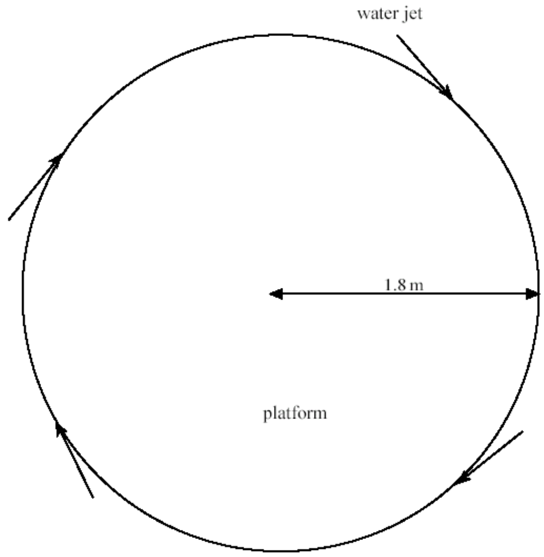
(a) Assuming that this power remains constant during the acceleration, calculate the energy supplied to the centrifuge by the motor.

(b) Hence estimate the moment of inertia of the rotating system.

(AQA past question)

14C.02.8

A rotating flower bed forms a novelty feature in the annual display of a horticultural society. The circular platform supporting the plants floats in a water tank and is caused to rotate by means of four water jets directed at the rim of the platform.



Each of the four jets exerts a tangential force of 0.60 N on the platform at a distance of 1.8 m from the axis of the rotation. The platform rotates at a steady angular speed, making one complete revolution in 110 s.

(a) Calculate:

(i) the total torque exerted on the platform by the four jets,

(ii) the power dissipated by the frictional couple acting on the rotating platform, showing your reasoning.

(b) When the water jets are switched off, all the kinetic energy of the loaded platform is dissipated as heat by the frictional couple and the platform comes to rest from its normal steady speed in 12 s.

(i) The kinetic energy of the loaded platform when rotating at its normal steady speed is 1.5 J.

Show that this value is consistent with your answer to part (a) (ii).

(ii) Calculate the moment of inertia of the loaded platform.

(AQA Past Question)

2. Thermodynamics

Tutorial 14C.03 First Law of Thermodynamics

AQA Syllabus

Contents

Before you tackle this tutorial, you might wish to look at Topic 13.

14C.031 The First Law of Thermodynamics

Thermodynamics is the study of heat flows and how they can be put to work. **Engines** work by converting heat energy into movement energy, which can then do useful jobs of work for us. We need to look at a couple of key words. A **system** is the object of interest whose behaviour we are monitoring in relation to its **surroundings**. A flask containing gas is a system; the water bath in which the flask is placed is its surroundings. The diagram below helps to show the idea (*Figure 14*):

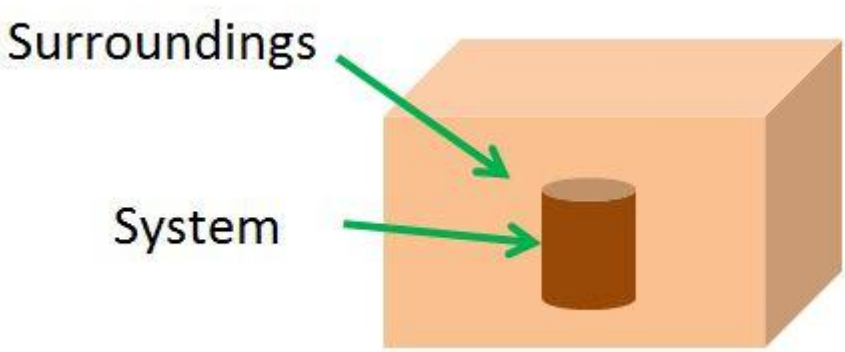


Figure 14 A thermodynamic system

The **Laws of Thermodynamics** were the results of work by nineteenth century physicists. Ironically the Second Law came before the First Law. Then a more fundamental law, the **Zeroth Law** was worked out.

In words the **First Law of Thermodynamics** is:

The change in internal energy of a system is equal to the sum of energy entering the system through heating and energy entering the system through work done on it.

We can write the first law in code:

$$\Delta Q = \Delta U + \Delta W$$

..............*Equation 52*

[ΔQ = heat entering the system; ΔU = increase in internal energy; ΔW - work done by the system]

The diagram here (*Figure 15*) explains the idea:

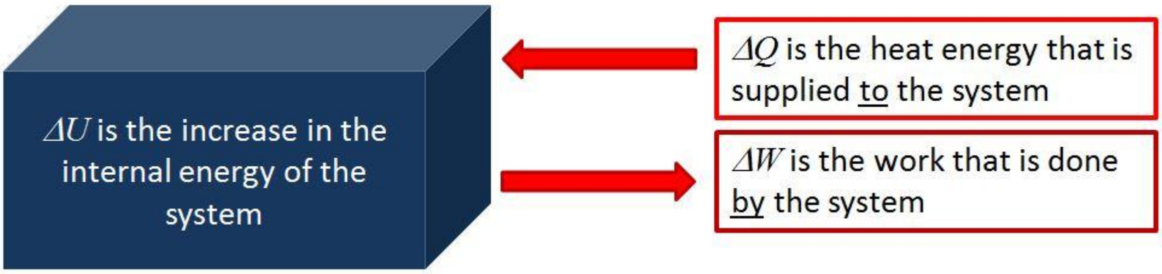


Figure 15 Explaining Figure 15

Worked Example A lump of lead of mass 0.50 kg is dropped from a height of 20 m onto a hard surface. It does not bounce but remains at rest. What are ΔQ, ΔW, and ΔU?
Answer ΔQ = **0 J** as zero heat is supplied to the system $\Delta U = mg\Delta h$ = 0.5 kg × 9.81 m s^{-2} × 20 m = **98 J** ΔW = **- 98 J** as work is done on the system rather than by the system.

If we compress a gas in a bicycle pump, we find it gets hot. Then we let the pump cool down, without releasing any gas.

Consider a cylinder of area A. A fluid is admitted at a constant pressure, p. It makes the piston move a distance, s (*Figure 16*).

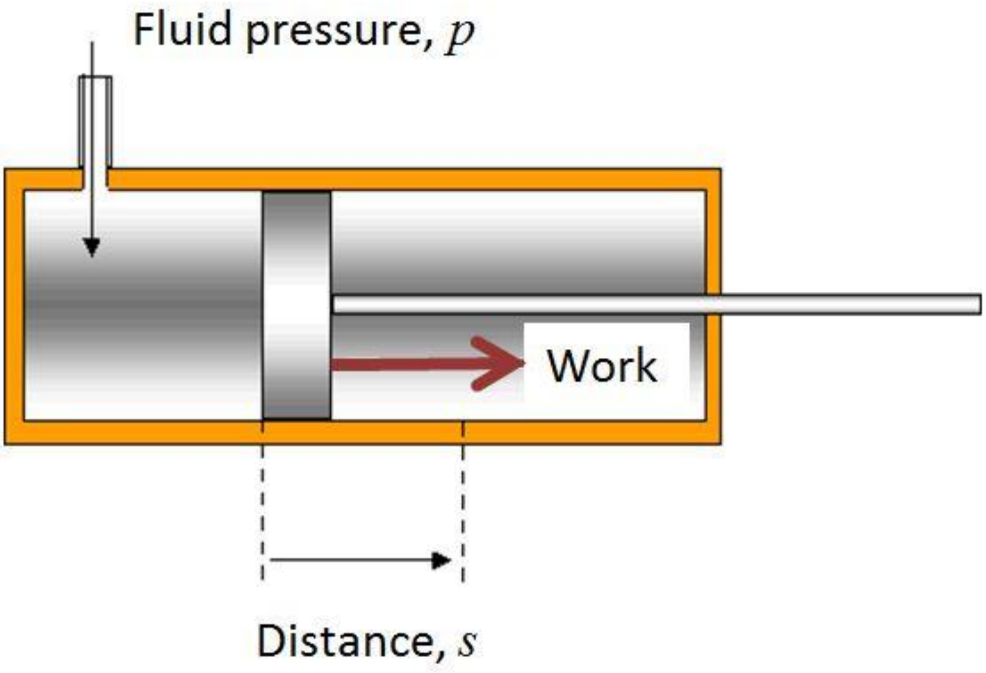


Figure 16 Work being done on a piston in a cylinder

We know that:

- Pressure (N m^{-2}) = force (N) ÷ area (m^2)

$$p = F/A \text{ .... Equation 53}$$

- Work done (J) = force (N) × distance moved (m)

$$W = F\Delta s \text{ .... Equation 54}$$

Therefore:

- Force (N) = pressure (N m^{-2}) × area (m^2)

$$F = pA \text{ .... Equation 55}$$

- Work done (J) = pressure (N m^{-2}) × area (m^2) × distance moved (m)

$$W = pA\Delta s$$ .... *Equation 56*

- Area (m^2) × distance moved (m) = change in volume (m^3)

$$\Delta V = A\Delta s$$ .... *Equation 57*

So, we can write:

Work done (J) = pressure (N m^{-2}) × change in volume (m^3)

In code:

$$\Delta W = p\Delta V$$ .... *Equation 58*

This can be shown in a graph (*Figure 17*):

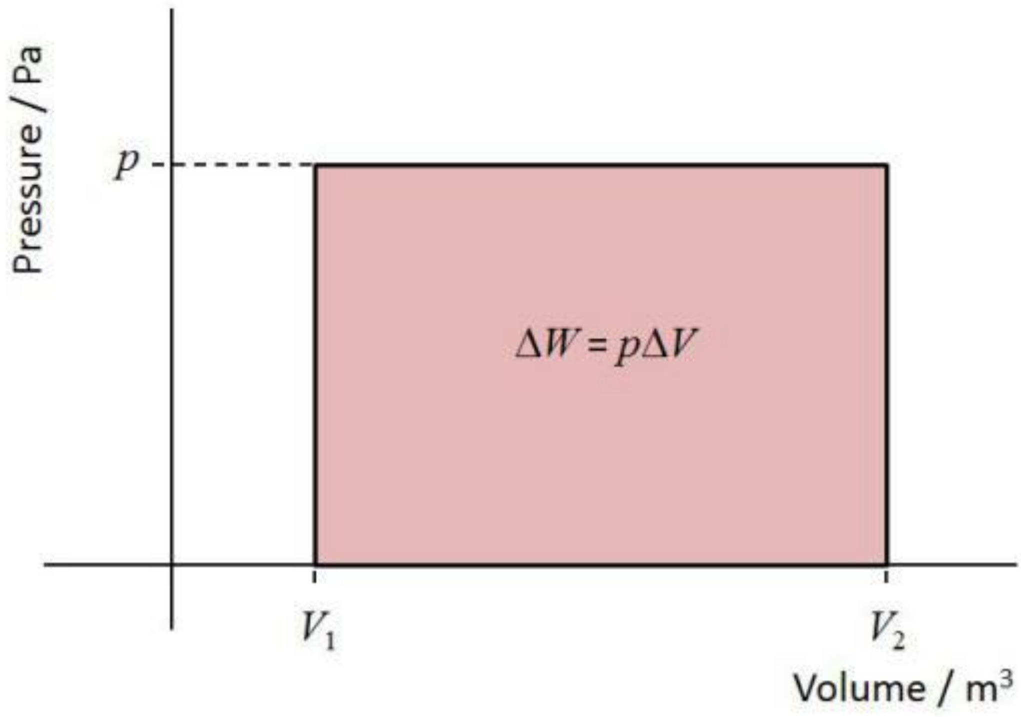


Figure 17 Change in volume at a constant pressure

The work done is the area under the graph.

14C.032 Isothermal Changes

In Topic 13 we saw that the behaviour of ideal gases is governed by the equation:

$$pV = nRT$$ ....*Equation 59*

If we keep the temperature the same, we can say that pV = constant, which you may remember as **Boyle's Law**. Keeping the temperature the same is called an **isothermal** compression or expansion. We can sketch a graph (*Figure 18*):

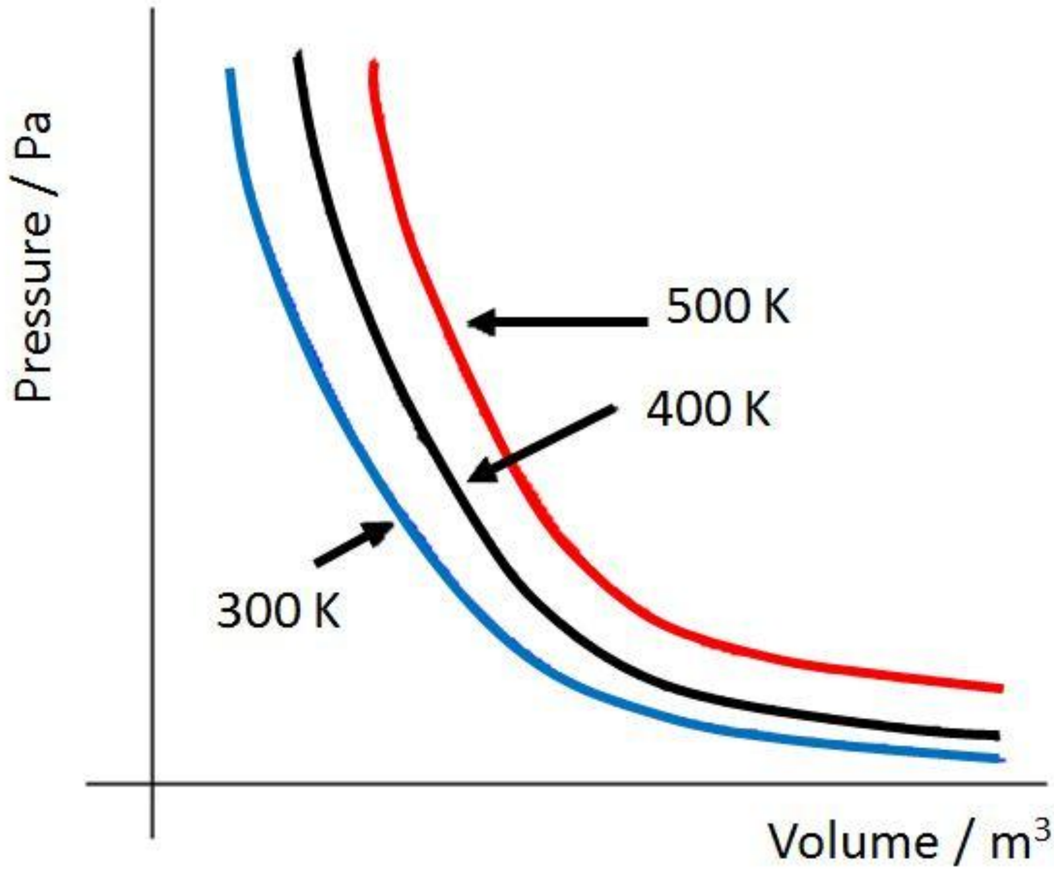


Figure 18 Isothermal compression or expansion

Each of the lines is called an **isothermal**, because the temperature is kept the same. We can make the compression and expansion of gases very nearly isothermal by pressing down on a bicycle pump very slowly, so that any heat generated can flow out very slowly. Similarly, we can allow the gas to expand very slowly so that the heat flow in in very slow.

For all isothermal processes:

- pV = constant and $p_1V_1 = p_2V_2$.
- $\Delta U = 0$ because the internal energy is dependent on temperature, which does not change.
- $\Delta Q = \Delta W$. If the gas expands to do work ΔW, and amount of heat ΔQ must be supplied

The process is a **reversible isothermal change** if the piston of the pump is allowed to expand after compression and follows **exactly** the same line on the graph that it did when being compressed and ends up in exactly the same place as when it started.

14C.033 Adiabatic Changes

A change where there is no heat flow in or out of a system is called **adiabatic**.

$$\Delta Q = 0, \text{ therefore } \Delta W = \Delta U$$

If you push the plunger of a bicycle pump in very rapidly and block off the end, you get an adiabatic process where the temperature rise of the gas is entirely due to the work done in compressing the gas. If a gas is allowed to expand without any heat energy being put in, the process is still adiabatic. The expansion occurs at the expense of the internal energy. The gas cools down.

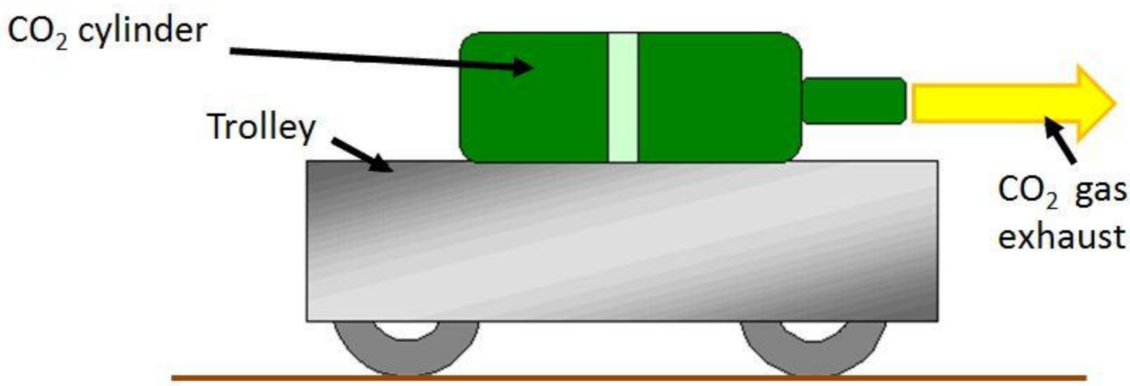


Figure 19 A trolley powered by a small carbon dioxide capsule

An example of this is a little rocket that can be made with a very small cylinder of carbon dioxide at high pressure (*Figure 19*). This is shown in the diagram above. The heat flow through the side is negligible compared with the energy loss that causes the drop in temperature as a result of the expansion of the gas. The cylinder gets so cold that frost forms on the outside, even though the room is warm.

We can use the gas laws to work out the temperature loss. A useful equation is:

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2}$$

.... *Equation 60*

<u>Worked Example</u> A small rocket powered trolley uses a CO_2 cartridge which contains 0.1 mol CO_2 gas. The volume of the cylinder is 15 × 10^{-6} m^3. The temperature of the compressed gas is 300 K. The gas is compressed to a pressure of 2.2 × 10^7 Pa. What is the temperature of the uncompressed gas assuming that 1 mol of gas occupies 24 × 10^{-3} m^3 at 1.01 × 10^5 Pa? (R = 8.3 J k^{-1} mol^{-1})
0.1 mole occupies 2.4 × 10^{-3} m^3 The equation to use is $$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2}$$ (2.2 × 10^7 Pa × 15 × 10^{-6} m^{-3}) ÷ 300 K = (1.01 × 10^5 Pa × 2.4 × 10^{-3} m^3) ÷ T_2 T_2 = (300 K × 1.01 × 10^5 Pa × 2.4 × 10^{-3} m^3) ÷ (2.2 × 10^7 Pa × 15 × 10^{-6} m^3) = **<u>220 K (-53 °C)</u>**

Temperatures are sometimes given in Celsius. They must be converted to Kelvin. Watch out for this bear trap.

We can look at the behaviour of a gas being compressed adiabatically. Remember no heat is allowed to enter or leave the system. Look at the light green line (*Figure 20*):

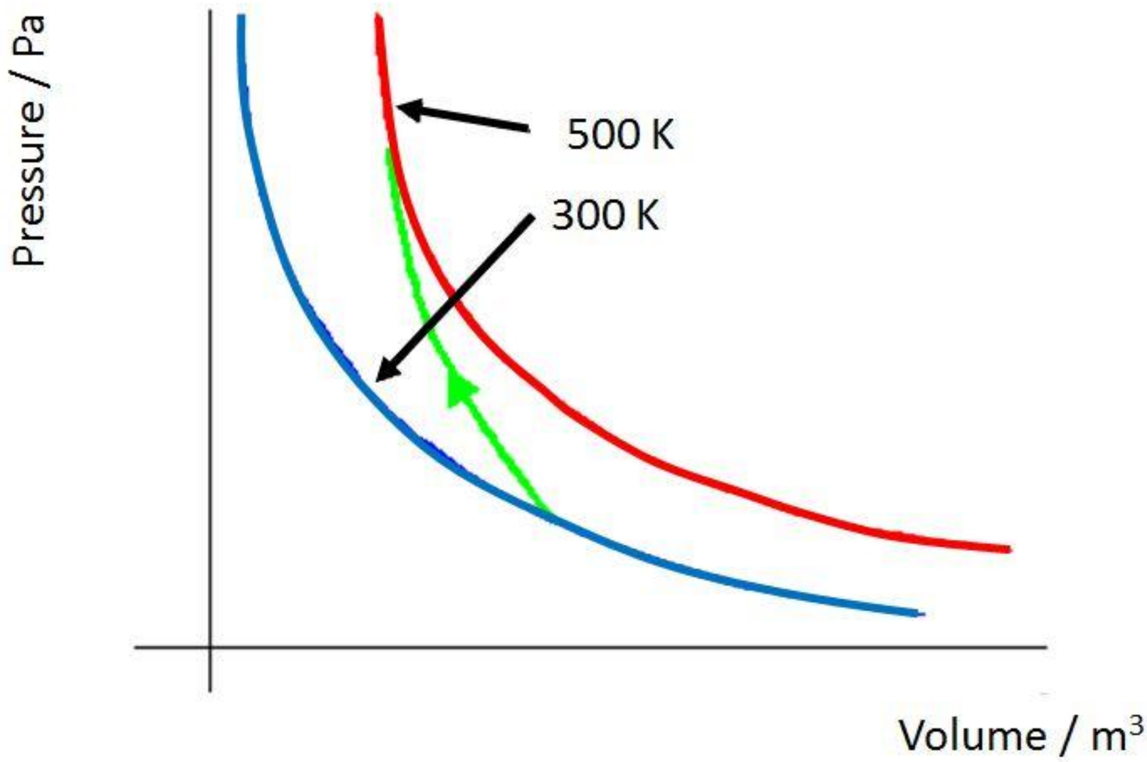


Figure 20 Adiabatic compression

The adiabatic line is steeper than the isothermal lines:

- At high pressure low volume, the adiabatic is at the value that you would expect for an isothermal at a high temperature; it has got hot.
- At low pressure, high volume the adiabatic line cuts the isothermal at a low temperature; the gas has become cool.
- The green line shows that as the volume decreases, the pressure rises, and *vice versa*.

- The equation for the adiabatic line is:

$$pV^{\gamma} = k$$

.... *Equation 61*

[k - constant; γ - ratio C_p/C_v]

- C_v is the energy needed to give unit temperature rise in 1 mole of gas where the volume is kept constant.
- C_p is the energy needed to give unit temperature rise in 1 mole of gas where the pressure is kept constant.
- For a monatomic gas, γ = 1.67.
- For a diatomic gas, γ = 1.40.
- For a polyatomic gas γ = 1.33.
- A more useful version of the equation is:

$$p_1 {V_1}^{\gamma} = p_2 {V_2}^{\gamma}$$

.... *Equation 62*

- Since $\Delta Q = 0$, $\Delta W = -\Delta U$

14C.034 Isovolumetric Processes

Isovolumetric processes occur at constant volume. We can show this on a graph that displays the isothermals as we have above (*Figure 21*).

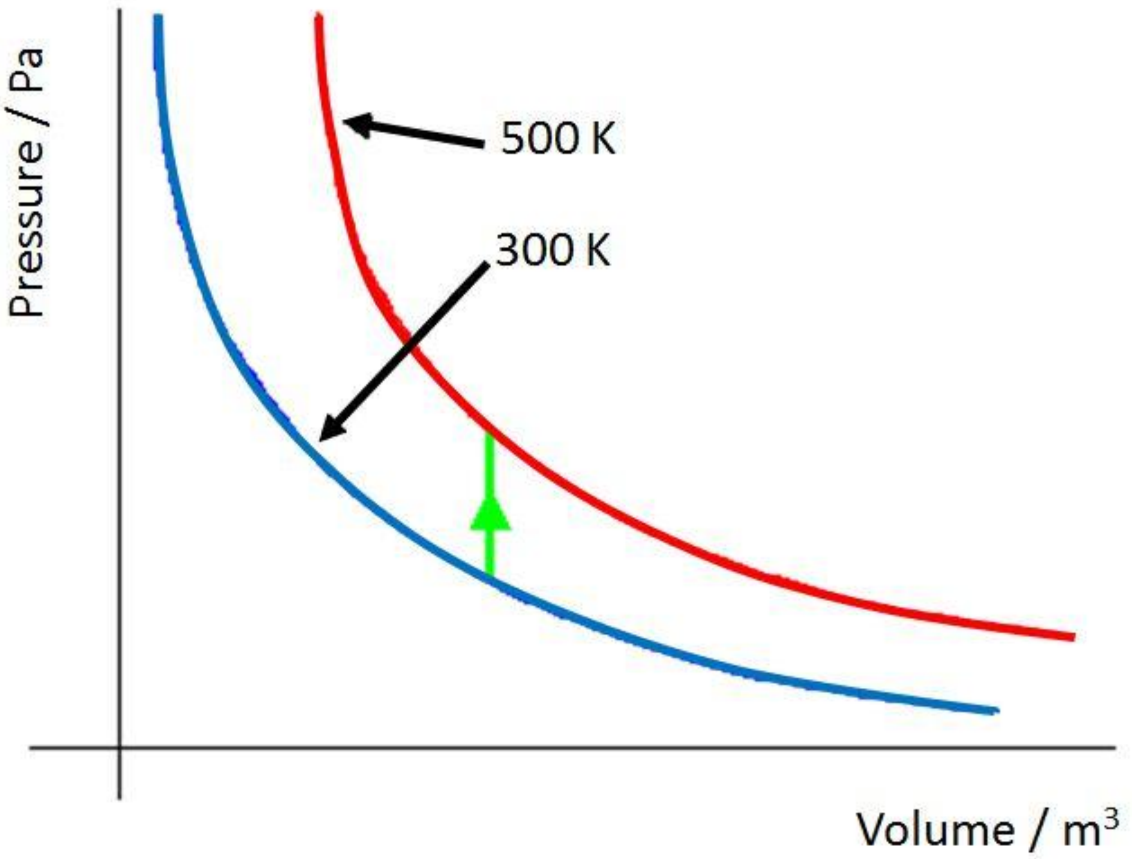


Figure 21PV graph of isovolumetric processes

- The process occurs at **constant volume**. The green line is vertical, showing that there is no change in volume.
- The pressure and temperature at constant volume is shown in *Equation 63*.

$$p_1/T_1 = p_2/T_2$$ .... *Equation 63*

- Since there is no change in volume, no work is done, so all heat entering the gas becomes internal energy.
- In other words, the green line shows the pressure increasing as energy is supplied.
- The end result of this is that the pressure goes up.

14C.035 Isobaric Processes

These happen at a constant pressure. The graph shows the idea (*Figure 22*):

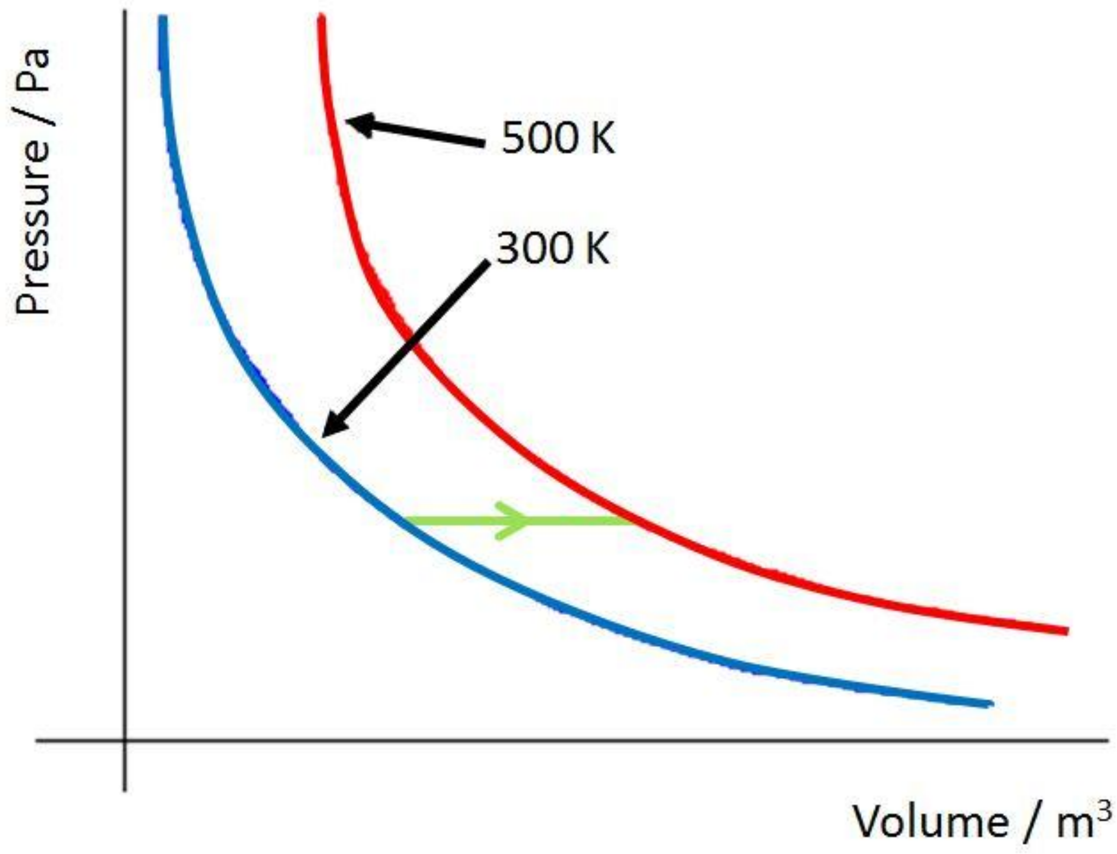


Figure 22 PV graph of isobaric change

- The process occurs at **constant pressure**. This is shown by the green horizontal line. The pressure stays the same.
- The equation is:

$$V_1/T_1 = V_2/T_2$$ .... *Equation 64*

- Some of the heat is used to increase the internal energy, the rest to do work.

Questions

Tutorial 14C.03

14C.03.1

What is internal energy?

14C.03.2

Some air in a bicycle pump is compressed so that its volume decreases and its internal energy increases. If 25 J of work are done by the person compressing the air, and if 20 J of thermal energy leave the gas through the walls of the pump, what is the increase in the internal energy of the air?

14C.03.3

What happens if we release the pump in Question 14C.03.2?

14C.03.4

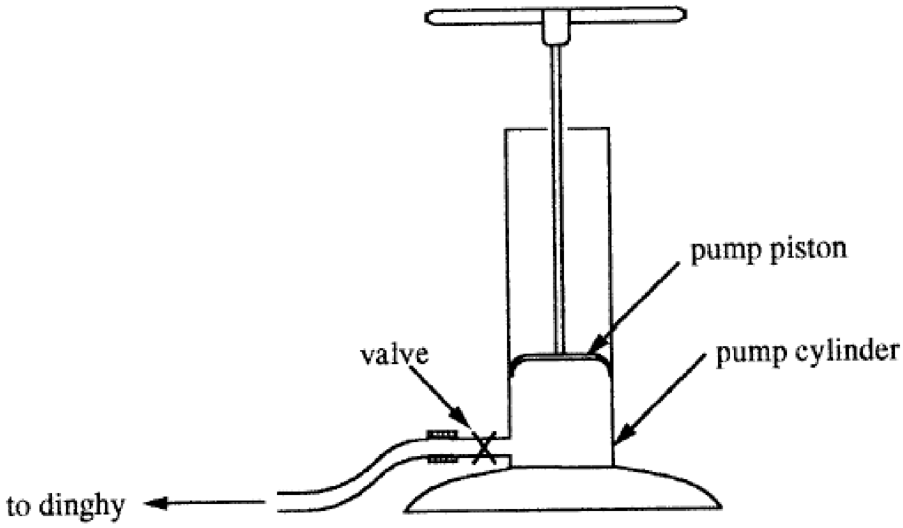
A cylinder has an area of 0.125 m^2. Steam is admitted at a pressure of 1.5 x 10^6 Pa. The piston moves a distance of 0.20 m. What work is done?

14C.03.5

Why can we assume that the behaviour of the gas is almost adiabatic?

14C.03.6

The diagram below shows a pump used to inflate a rubber dinghy. When the piston is pushed down, the pressure of air in the cylinder increases until it reaches the pressure of the air in the dinghy. At this pressure the valve opens and air flows at almost constant pressure into the dinghy.



(a)

The pump is operated quickly so the compression of the air in the cylinder before the valve opens can be considered adiabatic. At the start of a pump stroke, the pump cylinder contains 4.25×10^{-4} m^3 of air at a pressure of 1.01×10^5 Pa and a temperature of 23 °C. The pressure of air in the dinghy is 1.70×10^5 Pa. Show that, when the valve is about to open, the volume of air in the pump is about 2.9×10^{-4} m^3.

γ for air = 1.40

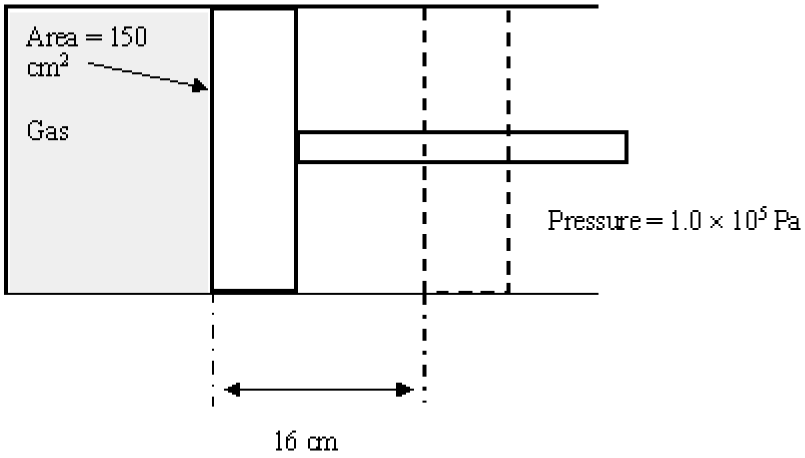
(b)

Calculate the temperature of the air in the pump when the valve is about to open.

AQA Past Question

14C.03.7

The diagram shows a sample of gas enclosed in a cylinder by a frictionless piston of area 150 cm^2.



When 300 J of energy is supplied to the gas, it expands and does work against a constant pressure of 1.0×10^5 Pa and pushes the piston 16 cm along the cylinder. Calculate:

(a) the work done by the gas

(b) the increase in internal energy of the gas.

(AQA Past Question)

Tutorial 14C.04 The P-V Diagram	
AQA Syllabus	
Contents	

14 C.041 P-V Diagrams

In Tutorial 14C.03, we look at the graphs of pressure against volume (**PV graphs**) for individual events. We are now going to take these a step further.

When a gas undergoes changes that will eventually return to its original state, it will go through a **cycle** of **processes**. The diagram (*Figure 23*) below shows an **ideal gas** undergoing some processes.

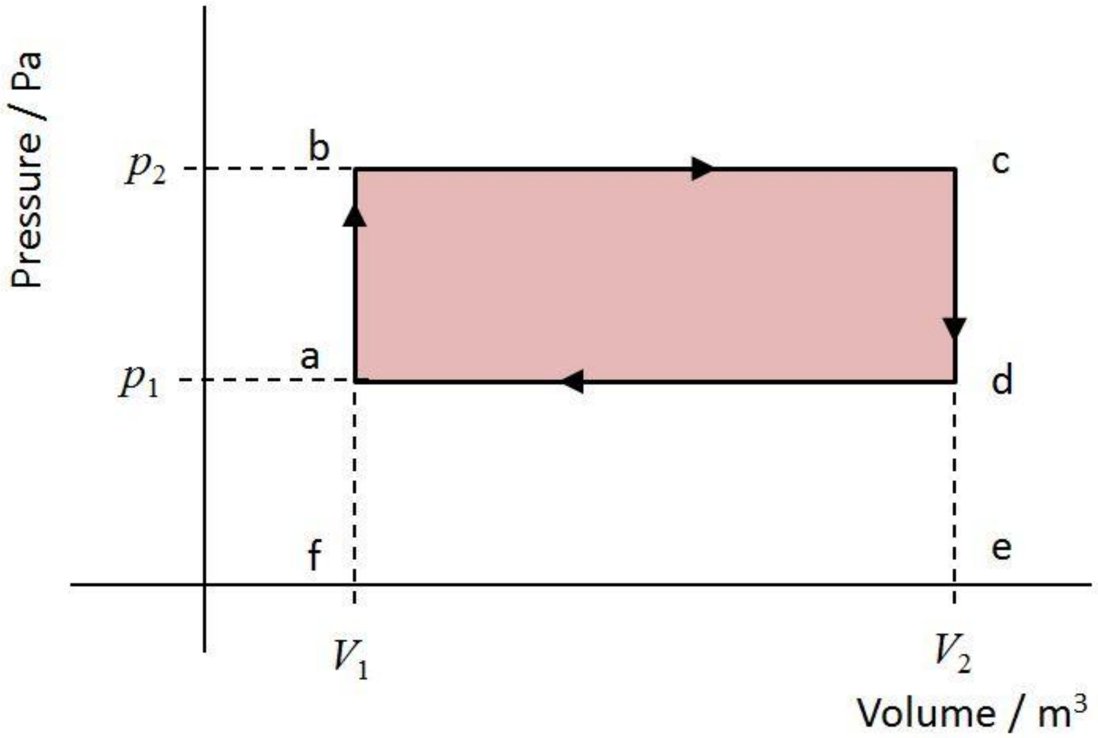


Figure 23 PV diagram for an ideal gas

The gas undergoes:

- **Isovolumetric** changes between **a** and **b**, and **c** and **d**. You may see this written as **isochoric** changes in some books.
- **Isobaric** changes between **b** and **c**, and **d** and **a**.

Let's analyse the changes and see what work gets done

- From **a** to **b**, there is no work done as it is an **isovolumetric** change.
- From **b** to **c** work is done by the gas as it expands. Work done is the area of the rectangle **bcef** = $p_2(V_2 - V_1)$
- From **c** to **d** there is no work done as the change is isovolumetric.
- From **d** to **a** work is done on the gas as it is compressed. Work done is the area of the rectangle **adef** = $p_1(V_2 - V_1)$
- The overall work done is the difference between the two areas, i.e. the area of the rectangle **abcd**.

The cycle diagrams are sometimes called **indicator diagrams** and are widely used by engineers looking at the work that can be got from an engine. The diagram (*Figure 23*) above is for an ideal gas, but there is a machine called a **Stirling Engine** that gives an indicator diagram that is very similar. Here is a picture (*Figure 24*) of the Stirling Engine which was invented in 1816.

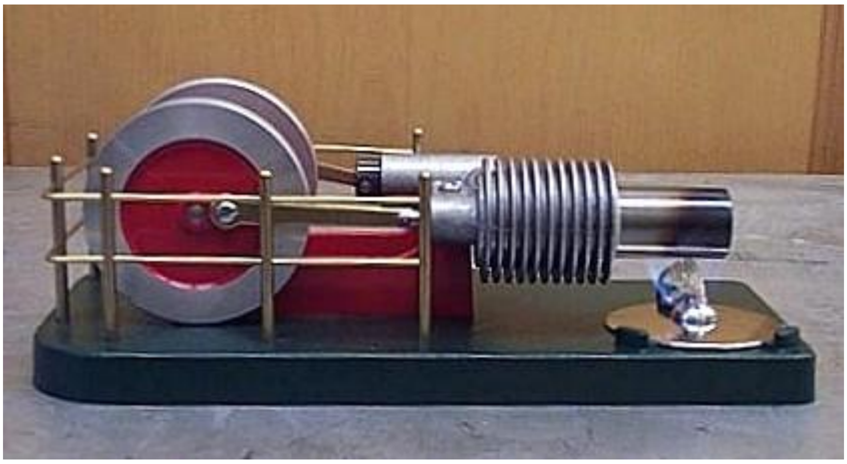


Figure 24 A Stirling engine

14C.042 The Stirling Engine

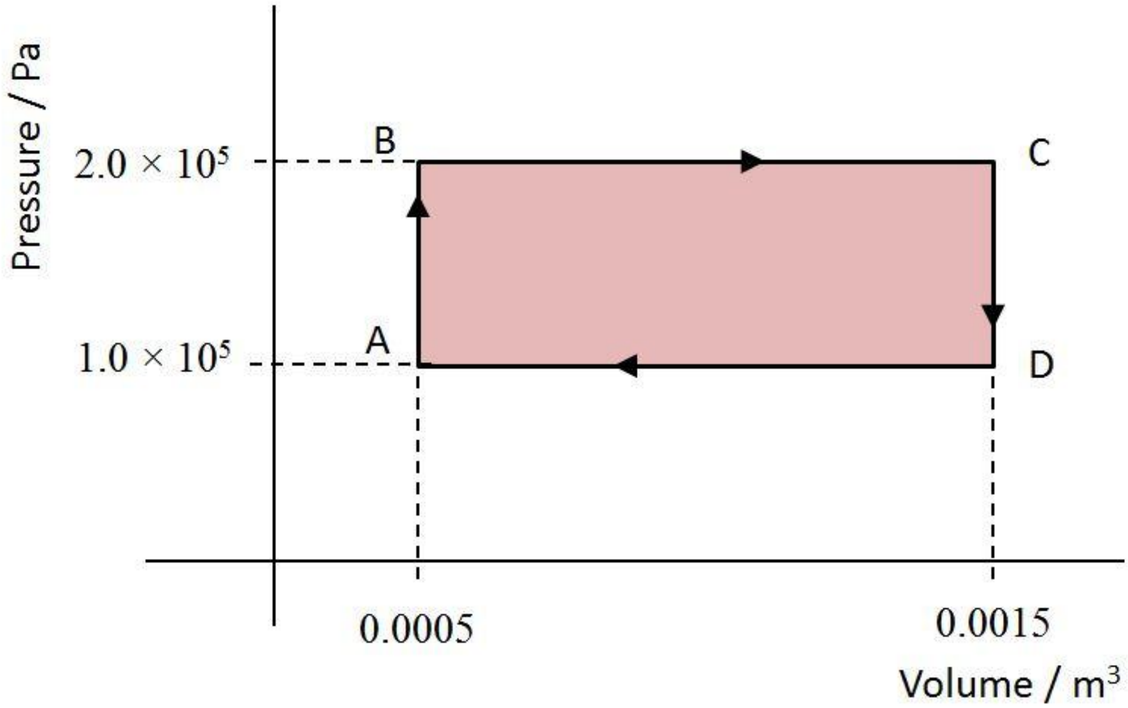


Figure 25 PV diagram for a Stirling engine

At point A air is in the cylinder at a pressure of 1.0×10^5 Pa and a temperature of 300 K. We always use absolute temperatures. We need to work out how many moles of gas there are in the cylinder.

Worked example Use the gas equation to find out how many moles there are in the cylinder at point A (*Figure 25*).
Answer $$pV = nRT$$ $n = pV/RT = (1.0 \times 10^5 \text{ Pa} \times 0.0005 \text{ m}^3) \div (8.3 \text{ J mol}^{-1} \text{ K}^{-1} \times 300 \text{ K})$ n = **0.0201 mol**

We can do an energy audit on the cycle. You are NOT expected to know about the molar heat capacity of a gas at constant volume, or the molar heat capacity of a gas at constant pressure. The table shows work done at various points about the cycle.

Point	***Heat supplied to gas /J***	***Work done on Gas /J***	***Increase in Internal energy /J***
A	125	0	125
B	700	-200	500
C	-375	0	-375
D	-350	+100	-250

We can describe what is happening:

- A to B 125 J is supplied to the gas raising its temperature at constant volume.
- B to C 700 J of heat is supplied, while the gas does 200 J of work on the surroundings.
- C to D 375 J is extracted from the gas to cool it at constant volume.
- D to A to return the gas to its starting point 100 J of work has to be done on the gas and 250 J are extracted from it so that the volume falls at constant pressure.

If we look at the indicator diagram, we can find the work done by the engine. It is the area of the pink rectangle.

Overall, 825 J are supplied as heat, while 725 J are extracted as heat, and lost to heat the surroundings. Of the work done, only 100 J is useful work done.

Therefore, we can write down the **thermal efficiency**:

Thermal efficiency = net work output ÷ heat input

Often, we multiply the resulting fraction by 100 to give a percentage. It is impossible to get anything more than 100 % efficiency, as that means that we would be creating energy.

And we can't, so there!

14C.043 Practical Efficiency

The thermal efficiency is not the actual efficiency of the engine. There will be frictional losses within the engine itself, reducing further the output available. The engineer can design the engine to be as efficient as possible:

- by considering the theory of how the gases behave as they expand and contract.
- by designing the engine so that friction is low, valves are gas tight, and that parts of the engine are manufactured with high precision. (Sloppy tolerances give rise to "slogger", which thwarts many attempts to increase efficiency.)

Although it is theoretically possible to get 60 % efficiency from a car engine, 30 % is more likely. Also, an initially highly efficient engine will lose efficiency as it wears out.

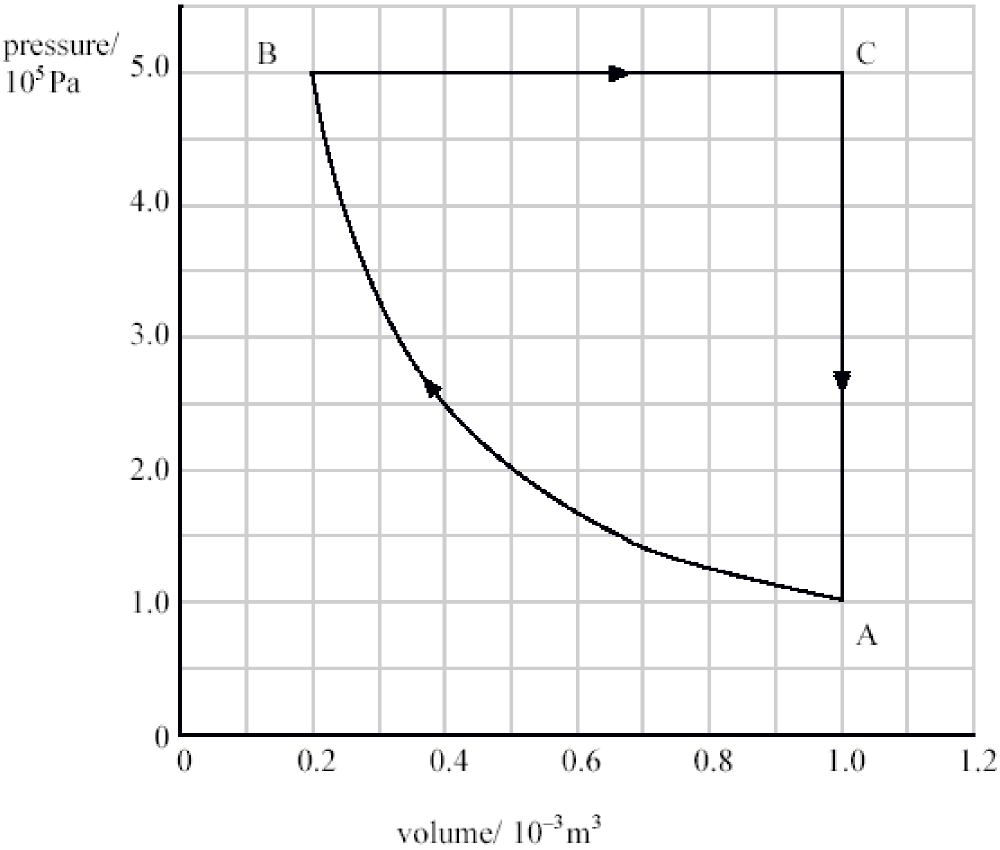
Getting useful work from heat is remarkably difficult.

Questions

Tutorial 14C.04

14C.04.1

The pV diagram shows a cycle in which a fixed mass of an ideal gas is taken through the following processes: A to B isothermal compression, B to C expansion at constant pressure, C to A reduction in pressure at constant volume.



(a) Show that the compression in process A B is isothermal.

(b) In which two of the three processes must heat be removed from the gas?

(c) Calculate the work done by the gas during process B to C.

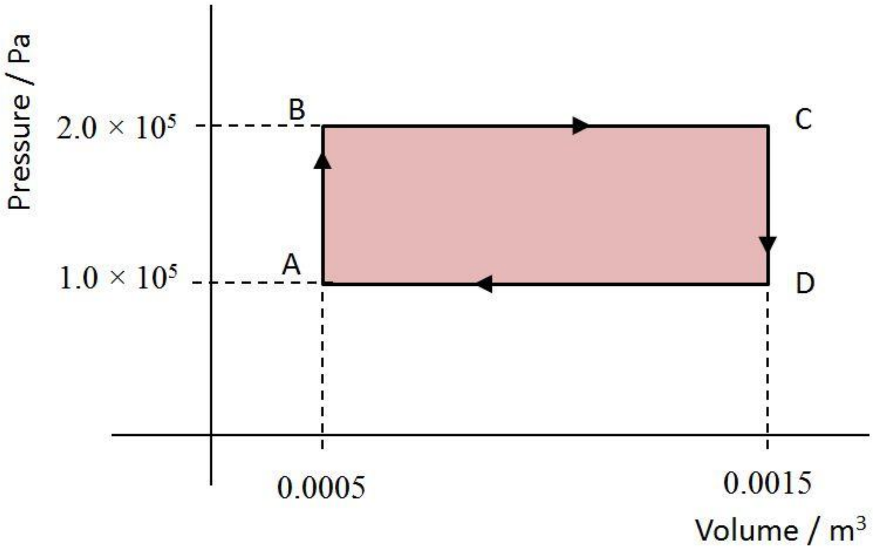
(d) The cycle shown in the diagram involves 6.9 × 10.2 mol of gas.

(i) At which point in the cycle is the temperature of the gas greatest?

(ii) Calculate the temperature of the gas at this point.

(AQA Past question)

14C.04.2



Use the graph above and the ideal gas equation to fill in the table below.

Point	***Pressure /Pa***	***Volume /m³***	***Temperature /K***
A	1.0×10^5	0.0005	300
B			
C			
D			

14C.04.3

Refer to the graph in Question 14C.4.2.

What is the work done?

14C.04.4

What is the thermal efficiency of the engine above in Question 14C.04.3?

14C.04.5

A single cylinder steam engine has an idealised indicator diagram as shown in Figure 1. Between A and B, the cylinder is connected directly to a source of high pressure steam. Between C and D, the cylinder is connected to the atmosphere.

Calculate the indicated power output of the engine when it is working at a rate such that one cycle takes 0.20 s.

(AQA Past Question)

Tutorial 14C.05 Internal Combustion Engines

AQA Syllabus

Contents

14C.051 Internal Combustion Engines

The internal combustion engine does away with the need for an external heat source. Fuel is burned within the engine to provide the heat that does the useful work. Generally, these engines use **fossil fuels** which are particularly concentrated forms of energy. We will look at the two most common types:

- The petrol engine which uses the **Otto Cycle**.
- The diesel engine.

The picture (*Figure 26*) shows a typical car engine.

Figure 26 This car engine uses the Otto cycle.

14C052 The Otto Cycle

The **four-stroke** Otto cycle is shown in the diagram (*Figure 27*):

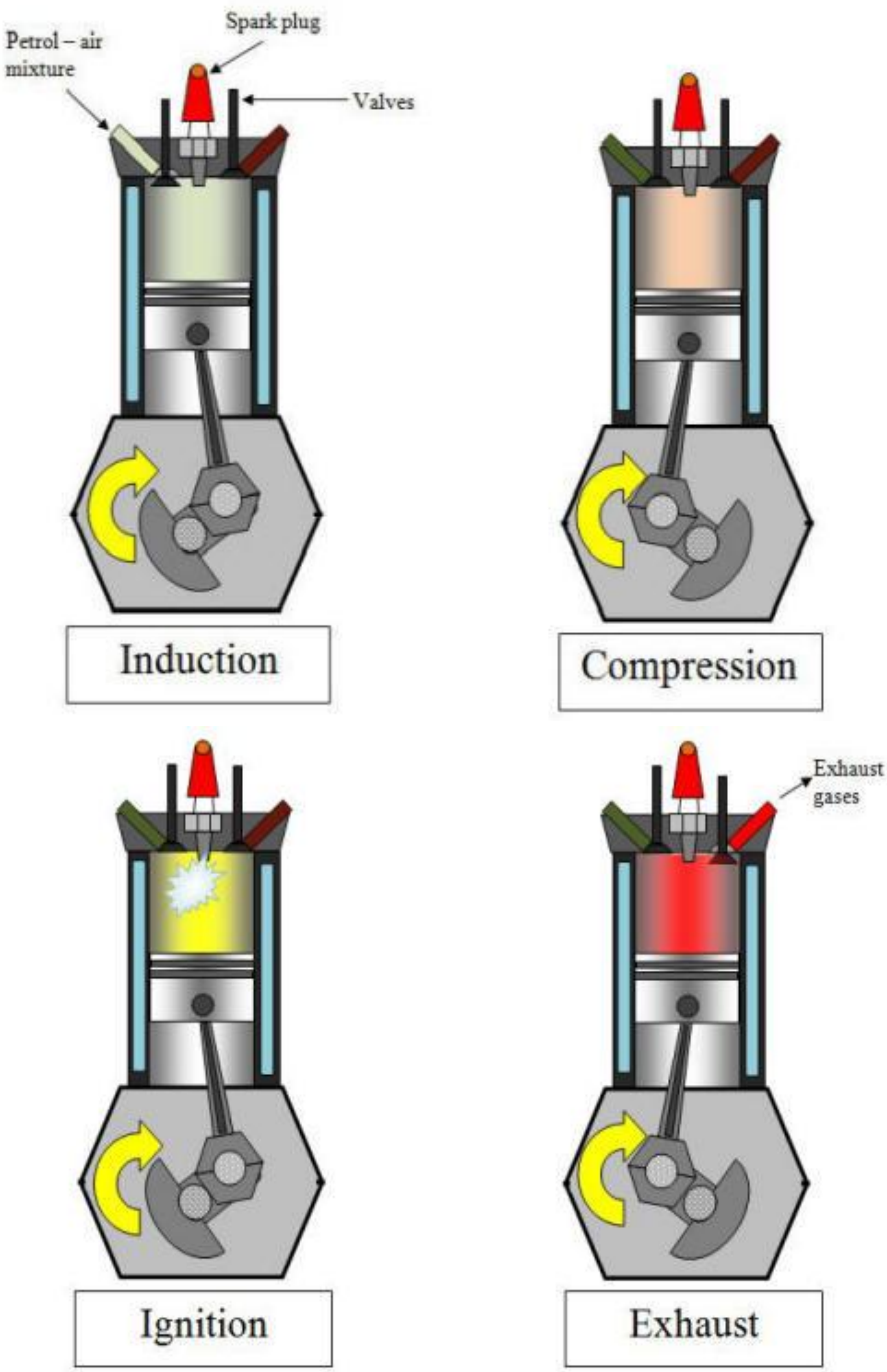


Figure 27 The Otto cycle

The indicator diagram for the Otto cycle is like this (*Figure 28*):

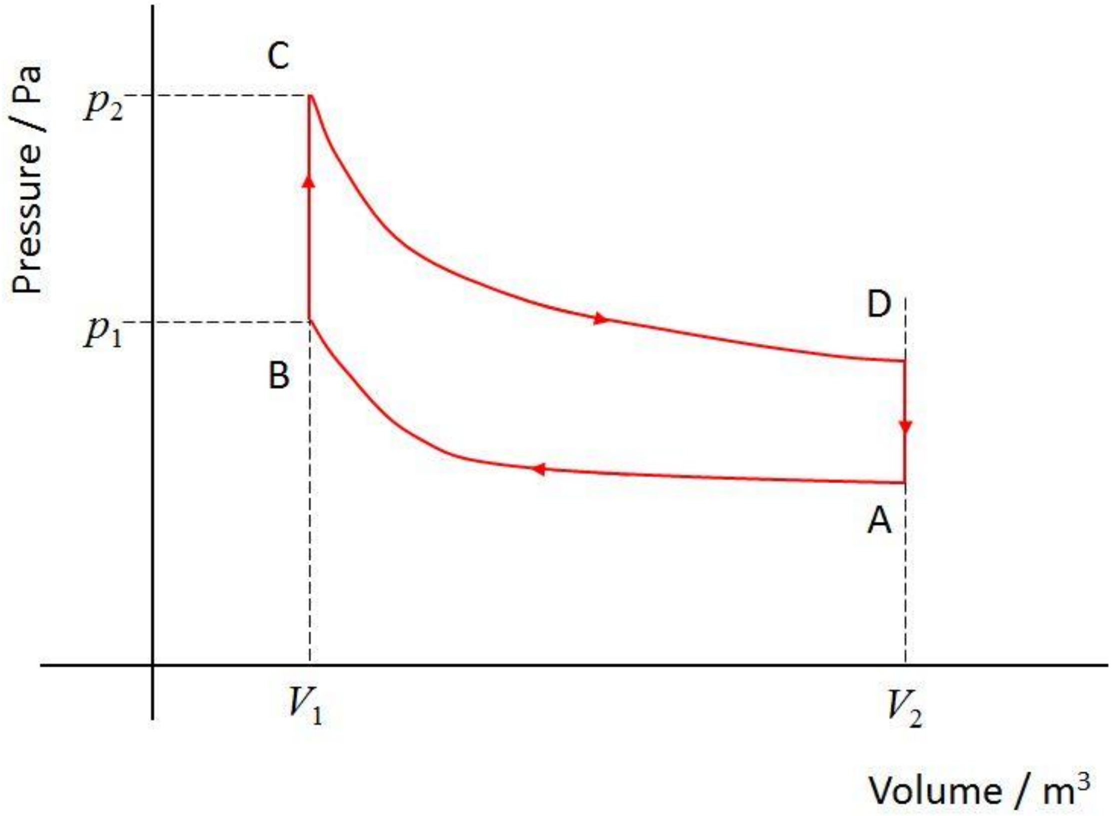


Figure 28 Indicator diagram for the Otto Cycle

Let's look at the cycle and link it to the indicator diagram:

1. The induction stroke takes place at A. Although in theory the pressure should be the same as atmospheric, in practice it's rather lower. The amount of petrol air mixture taken in can be increased by use of a supercharger.
2. A to B is the compression stroke. Both valves are closed. The compression is adiabatic, and no heat enters or leaves the cylinder.
3. Ignition occurs at C. The gases resulting from the ignition expand adiabatically, leading to the power stroke.
4. D to A the gas is cooled instantaneously.
5. At A the exhaust stroke occurs, and the gases are removed at constant pressure to the atmosphere.
6. Strange as it may seem, the piston does half a revolution at A. Actually, it's slightly more in practice, as the valve timing is more complex.

In practice the thermodynamics of a petrol engine are more complex:

- Fuel burns during the cycle, so the number of moles is not constant.
- The cycle takes place very quickly, so there is swirling of the gases. The kinetic energy of gases is not taken into account in these indicator diagrams.
- There are considerable temperature gradients, so we cannot deal with the gas as if it were constant temperature.
- Ignition takes a finite time and takes time to propagate through the fuel-air mix. Therefore, pressures will vary within the gas.

The picture shows a large petrol engine that was used in a war-time transport aeroplane. Each engine (*Figure 29*) had a capacity of 29 litres, with a power output of 750 kW (1000 PS).

Figure 29 A large petrol (Avgas) aero engine

The efficiency of a petrol engine can be increased by increasing the **compression ratio**. However, the heating of the gases can ignite the petrol prematurely. This **pre-ignition** is known as **knocking** or **pinking**. It can do a lot of damage to the engine.

14C.053 Diesel Cycle

The **Diesel cycle** (*Figure* 30) differs from the Otto cycle in that the induction stroke takes in only air. The air is compressed quite a lot so that it gets hot. The fuel is injected into the hot air and ignites. This produces the power stroke.

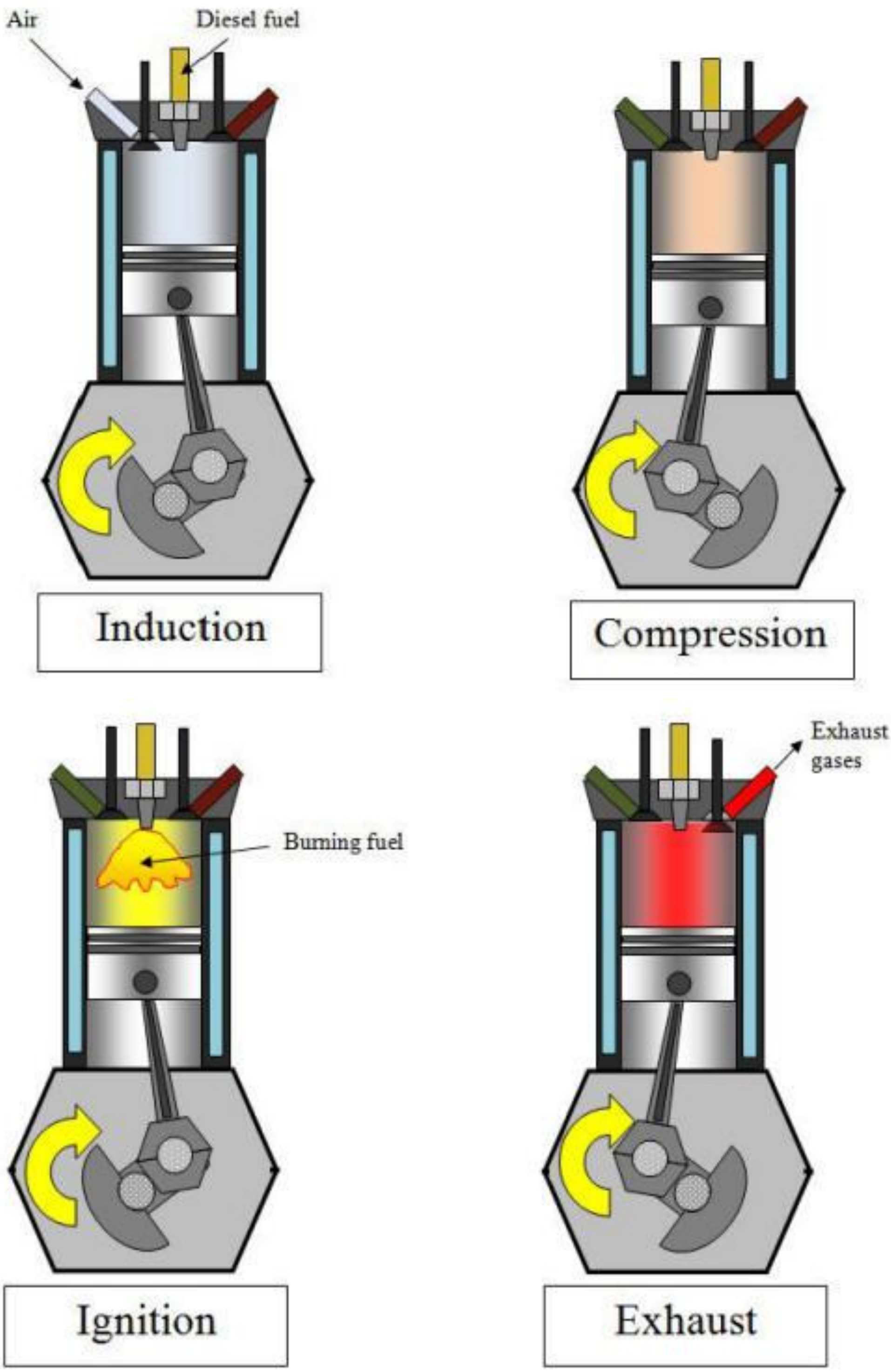


Figure 30 The Diesel cycle

The indicator diagram is quite different to that of a petrol engine (*Figure 31*):

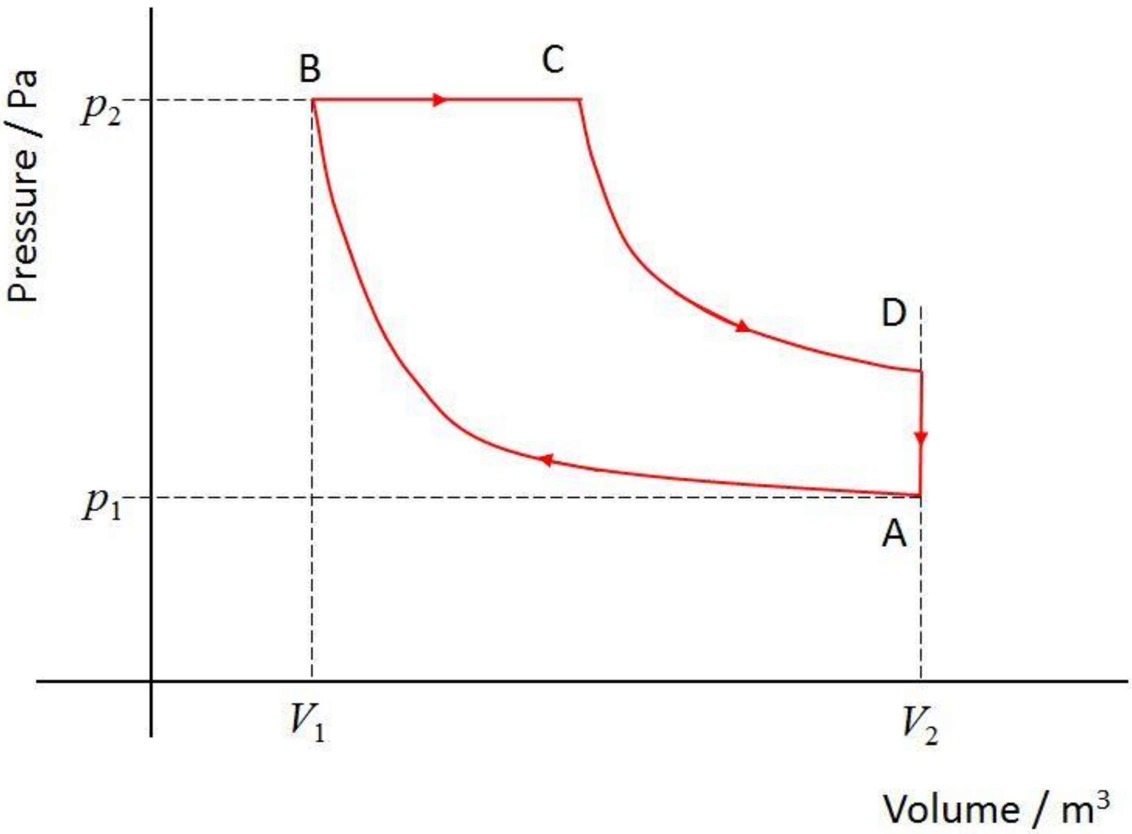


Figure 31 Indicator diagram for the Diesel cycle

Let's now look what happens in the indicator diagram:

1. The induction stroke takes air in ideally at constant volume, pressure and temperature.
2. The compression stroke takes place from A to B. The air is compressed adiabatically to about 1/20 of its original volume. It gets hot.
3. From B to C fuel is injected in atomised form. It burns steadily so that the pressure on the piston is constant.
4. From C to D the power stroke moves the piston down as adiabatic expansion takes place.
5. D to A cooling and exhaust occurs.

The diesel engine has a higher thermal efficiency than the petrol engine. However, it does have the disadvantage in that it is heavier. Also, the size of engine for a given power tends to be bigger. They also tend to be noisier, and incomplete combustion makes for considerable pollution. Complex exhaust systems have been developed to tackle this problem. However, they are expensive if they go wrong.

Several European countries have pledged to ban diesel cars by the year 2040.

However, diesels have been made lighter and more refined for luxury cars. Experiments with diesels for aircraft have been hugely successful. Jet A1 fuel (paraffin) costs 80 p a litre compared with Avgas (aviation petrol) at £2.00 a litre.

Figure 32 A piston-engine aircraft that runs on diesel fuel or Jet-A1

This aircraft (a Diamond Twinstar) uses two 2.0 litre diesels (of the same type as found in Mercedes cars, but with higher quality components). It can fly at 360 km/h, and flying at 150 km/h burns about 3 litres of fuel per hour. Rather more economical than a family saloon, but at 300 000 Euros not exactly a snip.

For either kind of engine, we can predict the power that the engine can give out by using a simple formula:

Power output = area of p-V loop × no of cylinders × number of cycles per second

A common bear trap is to forget that a single cylinder four stroke engine goes through each cycle **once every two revolutions**.

We can also work out the maximum energy that can be put into an engine by this formula:

Input Power = calorific value of fuel × flow rate of the fuel

The fuel for any engine has a **calorific value** which is the energy that can be got out of the fuel per unit mass. It is measured in joules per kilogram.

- For wood the calorific value is about 20×10^6 J kg^{-1}.
- For oil it is 42×10^6 J kg^{-1}.

In engineering articles, watch out for fuel flows in kg min^{-1} which need to be converted to kg s^{-1}. You do know how to do that, don't you?

14C.054 Turbines

Larger aeroplanes usually use **gas turbines**. Turbines work on the same principle as piston engines (suck, squeeze, bang, blow), but they are **rotary** engines rather than reciprocating engines. Therefore, they tend to be lighter than piston engines of equivalent power. *Figure 33* is an example (incomplete) on an old commercial aeroplane.

Figure 33 A gas turbine on an aircraft

The engine (a Rolls-Royce Dart) could produce 1100 kW, (1500 PS).

The picture below (*Figure* 34) shows the general layout of a gas turbine engine.

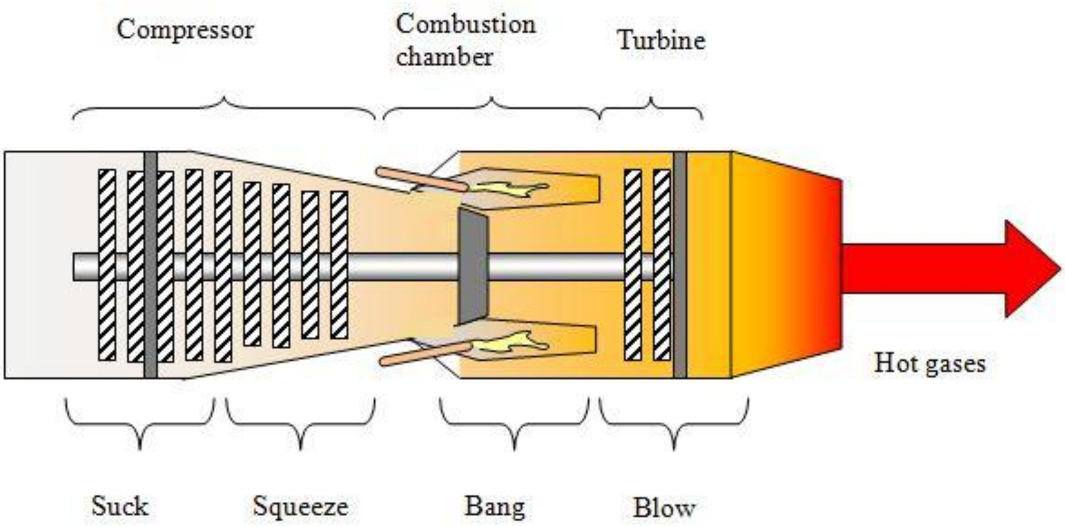


Figure 34 An aircraft gas turbine

Figure 34 shows a turbojet engine. It works on the same general principle of a diesel piston engine in the way that air is sucked in, compressed, fuel is ignited, and heated air rushes out of the back. In a **turbojet**, the heated air drives a turbine, which drives a compressor. In a **ramjet**, you don't have a turbine or a compressor, just a hollow tube. However, the aircraft has to be moving fast through the air for the engine to work.

The turbojet is less efficient than the **turbofan**, which is nowadays used widely in jet aircraft of all kinds (*Figure 35*).

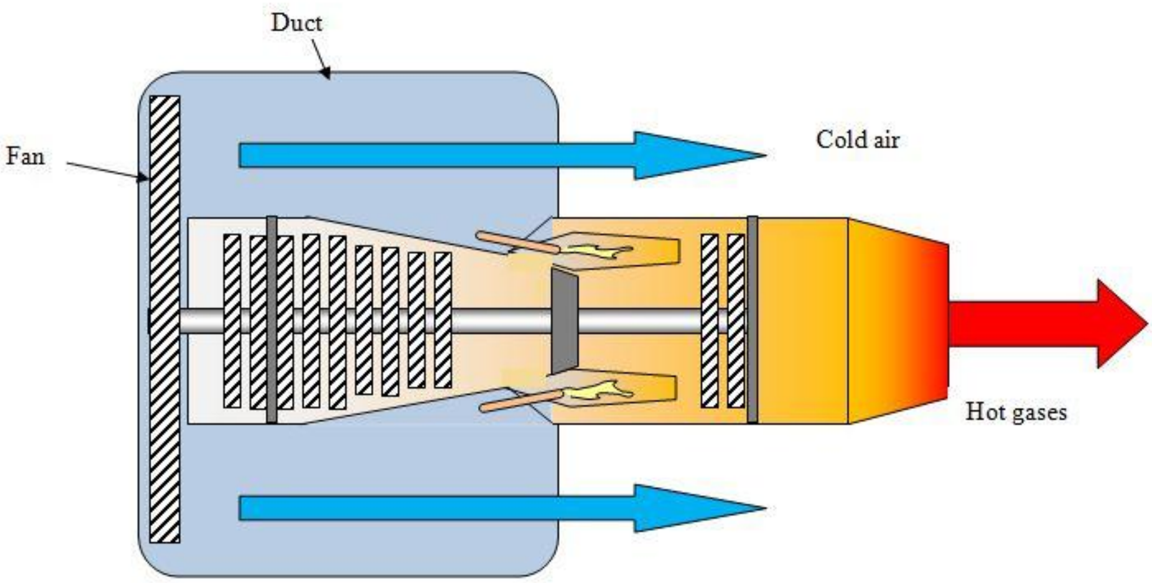


Figure 35 A turbofan

The jet engine drives a large fan in a duct, and large amounts of cold air are moved backwards without being heated. In the largest turbofans, the hot gases from the exhaust only contribute about 25 % of the total thrust. The fan at the front runs at a lower speed (3000 rpm) to the rest of the turbine (33 000 rpm), so there would be a reduction gearbox.

In a **turboprop** (*Figure 36*), the power is extracted from the hot air stream to drive a propeller. The hot gases from the back contribute about 5 % or less to the thrust.

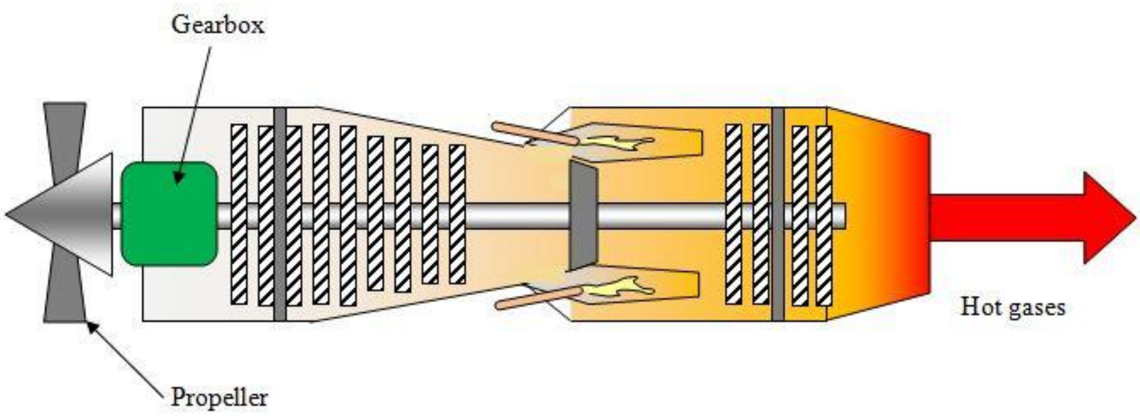


Figure 36 A turboprop engine

The turbine drives both the compressor and a propeller through a reduction gearbox. Typically, the propeller will turn at 2000 rpm, while the turbine spins at 30 000 rpm. Turbines like this can also be used to drive generators.

The advantage of a turbine like this is that it is more efficient at high power than a piston engine. If you are driving at 50 km/h, the engine in your car may be turning at 1000 rpm. If you speed up to 100 km/h, the engine will be turning at 2000 rpm. However, in an aeroplane, it's rather different. The propeller develops full thrust at 2500 rpm. At 1250 rpm it develops hardly any thrust at all. So, aircraft engines have to give all or nothing.

A turbine is very **inefficient** at low revs (which is why you don't find them in cars). It will gobble 50 kg of Jet A1 every hour when idling. At full power, it uses 150 kg every hour. Turbines also lose power at high altitudes where the air is thin. A turbo-charged piston engine retains its power.

The main disadvantage with turbines is that they are eye-wateringly expensive to buy and maintain. Machines that spin at 30 000 rpm have to be made to a high precision, requiring skilled craftsmen to make them. Any imbalance would shake the engine to pieces within seconds. A blade coming loose would smash every other blade off - the engineers call this "having a haircut". Another problem can be a "**surge**" or a **compressor stall**. The compressor stops compressing momentarily and the flame goes to the front of the engine with a loud bang, with a loss of thrust. A surging engine often leads to an emergency landing.

Another problem with turbines, often associated with compressor stalls but not always, is the **flame-out**. The fuel stops burning, leading to complete loss of power. The engine can be restarted, but it takes little imagination to see how such a situation can become extremely dangerous. Commercial pilots are trained (and tested) to deal with such eventualities using **simulators**.

14C.055 Wankel Rotary Engines

For smaller turbine aircraft, some engineers are suggesting the use of a **Wankel rotary engine**, designed by the German engineer Felix Wankel (1902 - 1988), which acts in a similar way to a piston petrol engine (*Figure 37*). These have the advantage of being more efficient than a piston engine, as well as being much lighter, but are a lot less expensive than a turbine. They are less prone to being over-revved, which can do catastrophic damage to a piston engine.

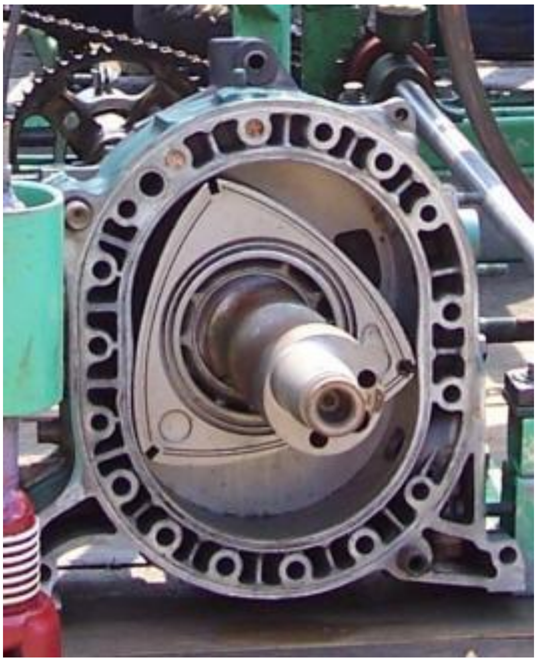


Figure 37 A Wankel rotary engine (Photo by J Lyon, Wikimedia Commons)

T

he picture (*Figure 37*) shows the rotor of a Mazda 1.3 litre Wankel engine that is found in some of their sports cars. Some aircraft manufacturers are planning to use these in light aircraft. In larger aircraft, a Mazda engine of capacity 2.6 litres is being considered, as these can safely produce 525 kW (700 PS).

The limiting factor with Wankel Rotary engines is the seal at the tip of each rotor, which tends to wear. Also, the engines discussed here run on petrol, which is expensive. The "holy grail" is to get a Wankel engine that runs on Jet A1. This is more difficult as diesel engines need a high compression, although there has been some success with diesel Wankel engines.

14C.056Testing Engines

Before any engine is put on the market, it has to be thoroughly tested. Nobody wants an engine that is going to fail in use. Nor do they want one that gobbles the fuel. There is nothing more useless than a broken-down car. In an aeroplane, you cannot pull off and stop behind a cloud; there is only one way - down. So various tests are done on engines, of which we will look at a few. One of the most common is **Useful Power**.

You will have seen that many engines have their power quoted as **brake horsepower** (bhp). This has been used by engineers for at least two hundred years. At its crudest, it is a comparison with the power you can get out of a horse, which had been the common form of motive power for many centuries. However a more scientific test was needed, and the diagram (*Figure 38*) below shows the kind of set up, called a **dynamometer**.

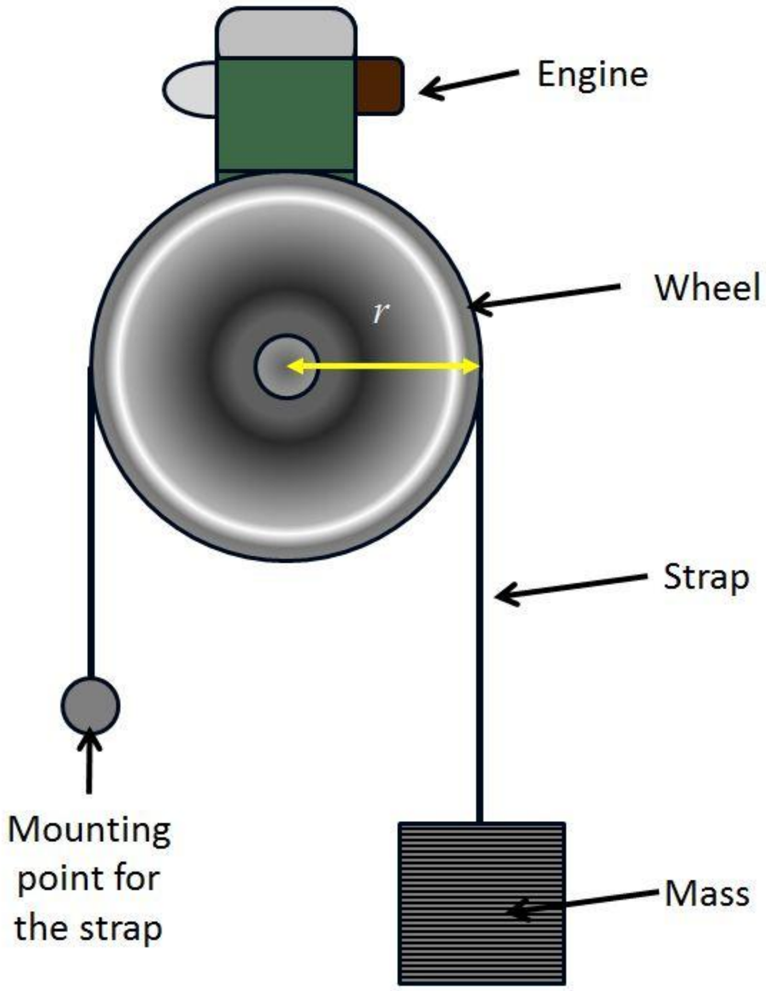


Figure 38 A dynamometer

The mass and strap act as a brake because they provide a frictional couple against the rotation of the engine.

$$\tau = mgr$$ .... *Equation 65*

The power produced by the engine is given by the formula:

$$P = \tau\omega$$ .... *Equation 66*

[P - power (W); τ - torque (N m); ω - angular velocity (rad s^{-1})]

Originally 1 brake horsepower worked out at 746 W; now it is considered as 750 W. It is often given the shorthand **PS** ("*Pferdstärke*", German for "horsepower").

The method above, although simple, has a disadvantage in that a lot of heat is generated. Although the principle is much the same, the test beds for engines are much more sophisticated. They can be:

- hydraulic with the engine driving a pump;
- electrical with the engine driving a generator into a load.

The picture (*Figure 39*) below shows an engine test bed:

Figure 39 An engine test bed (Photo Aniketdp.mech, Wikimedia Commons)

This picture (*Figure 40*) shows a water pump that an aircraft engine manufacturer uses to test engines before they are reinstalled into aircraft. It mimics the loads experienced by the engines in flight.

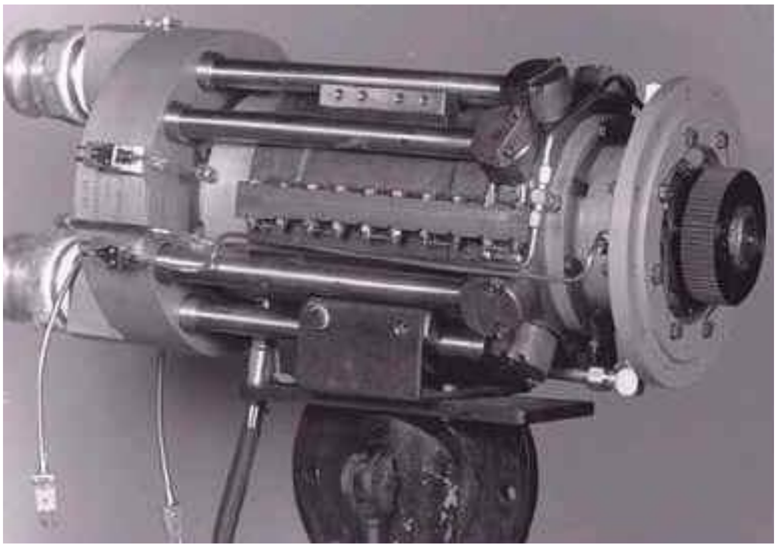


Figure 40 A water pump used to test aircraft engines

The useful power that can be got from an engine is always less that the power worked out from an indicator diagram. This is because there is friction within the engine. The power needed to overcome friction is the **friction power**:

friction power = indicated power - brake power

The answer you worked out in the Question 14C.05.3 shows that a lot of power is used to overcome friction. It is dissipated as heat. Oil **lubrication** is essential in such an engine:

- It reduces friction.
- It takes away the heat produced by friction.

Without lubrication the engine would rapidly seize up.

Aircraft engines are subject to rigorous testing, as catastrophic failure can be disastrous. On aerodromes, flocks of birds can be a nuisance, since an aeroplane running into a flock will cause serious damage, and not just to the birds. Recently an airliner climbing from a New York airport ingested birds into both engines, which stalled and failed completely. The pilot glided his 56 tonne aeroplane to a successful landing in the Hudson River. Thanks to his skill and training, everybody got off safely. The aeroplane was fished out of the river and is now in a museum. Other similar accidents have not ended so happily.

Sometimes engineers test the engines **to destruction**, including firing (dead) chickens at them, or deliberately shearing a blade at high speed. Such tests are very expensive, so good results are essential.

14C.057 Engine Efficiency

The **indicated** or **thermal efficiency** is given by:

thermal efficiency = indicated power ÷ power input from the fuel

As we have seen there are mechanical losses in an engine. The **mechanical efficiency** of an engine can be defined as the ratio of the output power to the indicated power or workable power. The output power is the power we can get from a dynamometer.

mechanical efficiency = output power ÷ indicated power

As the engine runs faster, the power absorbed in overcoming friction increases, so the mechanical efficiency falls away. We can see this in the graph below (*Figure 41*):

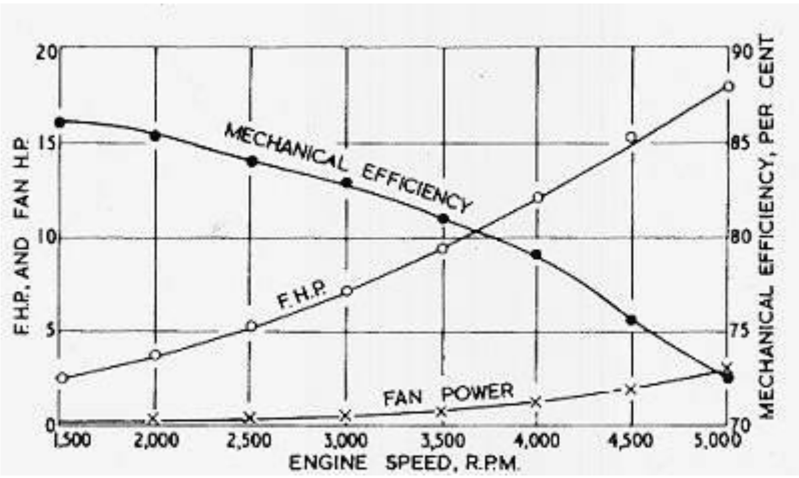


Figure 41 The mechanical efficiency of an engine falls as the speed increases

The frictional power increase almost mirrors the decrease in mechanical efficiency.

The **overall efficiency** is the fraction of the input power of the fuel that is delivered as useful power:

Overall efficiency = output power ÷ input power of the fuel.

The overall efficiency of internal combustion engines is not very good, with even the best being about 40 %.

Questions

Tutorial 14C.05

14C.05.1

Test-bed measurements made on a single-cylinder 4-stroke petrol engine produced the following data:

- mean temperature of gases in cylinder during combustion stroke 820 °C
- mean temperature of exhaust gases 77 °C
- area enclosed by indicator diagram loop 380 J
- rotational speed of output shaft 1800 rev min^{-1}
- power developed by engine at output shaft 4.7 kW
- calorific value of fuel 45 MJ kg^{-1}
- flow rate of fuel 2.1×10^{-2} kg min^{-1}

(a) The rate at which energy is supplied to the engine.

(b) The indicated power of the engine.

(c) The thermal efficiency of the engine.

(AQA Question, adapted)

14C.05.2

An engine gives out a torque of 250 N m at 3300 rpm. What is its power in watts and PS?

14C.05.3

In Question 1 you worked out that the indicated power of an engine was 5700 W. The power available at the output shaft is 4.7 kW.

What is the power dissipated in overcoming frictional losses in the engine?

What fraction is this of the indicated power?

14C.05.4

What is the mechanical efficiency of the engine in Question 14C.05.3?

14C.05.5

In Question 14C.05.1 you worked out that the power gained from the fuel of the engine was 15.8 kW. If the power output is 4.7 kW, what is the overall efficiency?

Tutorial 14C.06 Second Law of Thermodynamics and Heat Pumps	
AQA Syllabus	
Contents	
14C.061 Second Law of Thermodynamics	14C.062 The Carnot Cycle
14C.063 Heat Pumps	14C.064 Other Laws of Thermodynamics (Extension)

14C.061 Second Law of Thermodynamics

Before you tackle this tutorial, you might wish to look at Topic 13 – Thermal Physics

The **Second Law of Thermodynamics** states that it is impossible for any heat engine to be 100 % efficient:

No process is possible which results in the extraction of an amount of heat from a reservoir and its conversion to an equal amount of mechanical work.

There are different ways of stating this. The **Kelvin** statement of The Second Law is:

It is impossible to devise a cyclically operating device, the sole effect of which is to absorb energy in the form of heat from a single thermal reservoir and to deliver an equivalent amount of work

The **Kelvin-Planck** statement of the Second Law is:

It is impossible to devise a cyclically operating thermal engine, the sole effect of which is to absorb energy in the form of heat from a single thermal reservoir and to deliver an equivalent amount of work

The Clausius statement says:

Heat can never pass from a colder to a warmer body without some other change, connected therewith, occurring at the same time.

The theory behind these statements is that **entropy** increases. In other words, all processes tend towards chaos (which might explain my physics lessons when I worked in schools). If you drop a pack of cards, they will scatter and the chances of their landing in a meaningful order are very small indeed.

Heat is work and work's a curse,

And all the heat in the Universe

Is gonna cool down.

That will mean no more work,

And there'll be perfect peace.

That's entropy, man!

[Michael Flanders and Donald Swan]

Most energy is lost to the surroundings as **low grade heat**. We can show this in the diagram below:

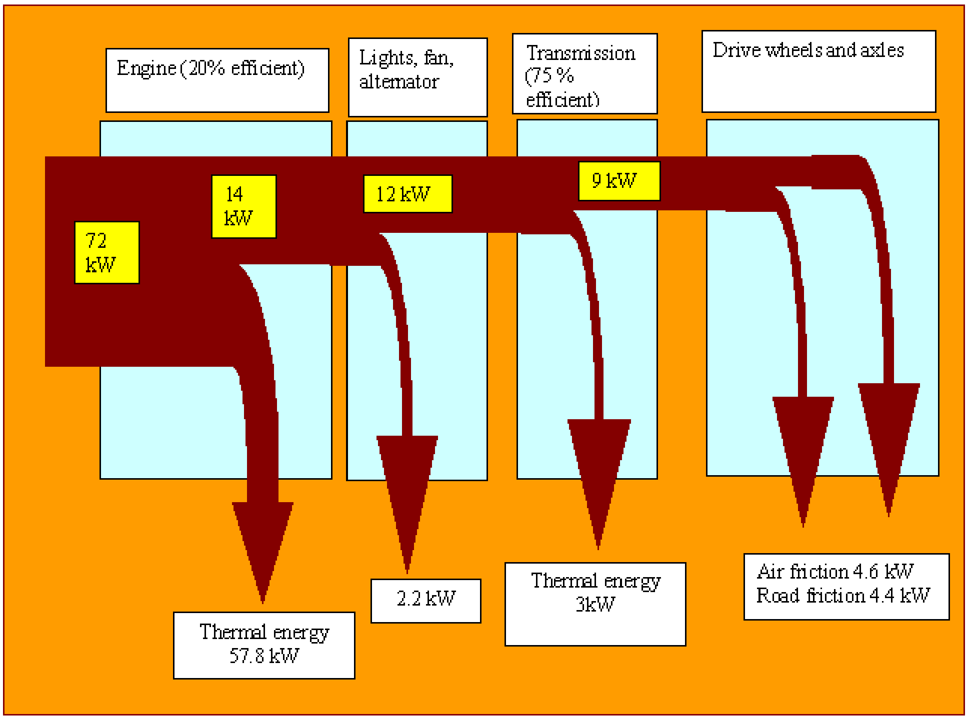


Figure 42 A Sankey diagram

In this diagram, called a **Sankey Diagram**, we can see that of 72 kW of power from the fuel, only 9 kW are used in actually driving a car along a road. The rest is lost as low grade heat. As we said before, getting energy out of heat is remarkably difficult.

All heat engines work by extracting mechanical energy from a **temperature gradient**. A heat engine has to operate between the hot reservoir and the cold reservoir to satisfy the Second Law of Thermodynamics. Heat flows from hot to cold, never the other way round:

Heat won't pass from a cooler to a hotter.

You can try it if you like,

But you far better notta,

Because the cold in the cooler

Will get hotter as a ruler,

And that's a physical law!

[Michael Flanders and Donald Swan]

We can show the heat flowing from a **hot reservoir** through a heat engine to a **cold reservoir** (*Figure 43*).

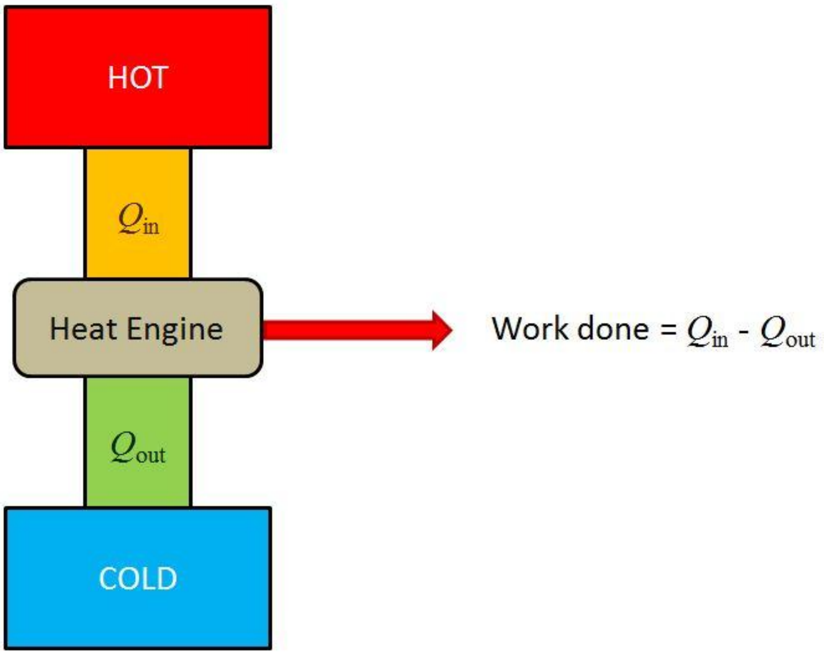


Figure 43 Heat energy flowing from hot to cold.

All heat engines give up their energy to a **cold reservoir**. We can define the terms used on the diagram (*Figure 43*):

- Q_{in} = the heat flow from the hot reservoir to the engine
- Q_{out} is the heat flow from the engine to the cold reservoir.
- The work done by the heat engine is the difference between Q_{in} and Q_{out}.

Therefore:

$$W = Q_{in} - Q_{out}$$ .... *Equation 67*

We can write down an efficiency relationships from *Equation 67*:

$$\text{Efficiency} = \frac{W}{Q_{in}} = \frac{Q_{in} - Q_{out}}{Q_{in}}$$

...............*Equation 68*

An ideal heat engine takes a quantity of heat Q_{in} from a hot reservoir of temperature T_H and sends a quantity of heat Q_{out} as waste to a cold reservoir of temperature T_C. It can be shown that:

$$\frac{Q_{out}}{Q_{in}} = \frac{T_C}{T_H}$$

.... *Equation 69*

We can rewrite the efficiency equation:

$$\text{Efficiency} = 1 - \frac{T_C}{T_H}$$

.... *Equation 70*

This can be rearranged to give us a useful relationship:

$$\text{Efficiency} = \frac{T_H - T_C}{T_H}$$

.... *Equation 71*

The temperature must always be in **Kelvin**. If we set T_C at 0 K, we could have a heat engine that was 100 % efficient, but as we can't get down to 0 K, forget it! However, we can make heat engines more efficient by making the difference between that hot reservoir and the cold reservoir as big as possible. In a power station, the steam coming from the boiler is at about 400 °C, while for the cold reservoir, water at about 10 °C is used.

Note that some text books use the code η for efficiency. The strange looking symbol η is "eta", a Greek letter long 'ē'. Some text books also refer to this as the **Carnot efficiency**.

14C.062 The Carnot Cycle

The most efficient heat cycle is called the Carnot Cycle. It consists of **two isothermal processes** and **two adiabatic processes**.

An **isothermal process** is one where the pressure in an ideal gas can be expressed in terms of the volume:

$$P = \frac{nRT}{V}$$

.... *Equation 72*

An **adiabatic condition** arises when the following relationship applies for an ideal gas:

$$pV^{\gamma} = \text{constant}$$ .... *Equation 73*

The power, γ, has a value of 5/3 or 1.67 for an ideal gas.

The idea is shown in the diagram below (*Figure 44*):

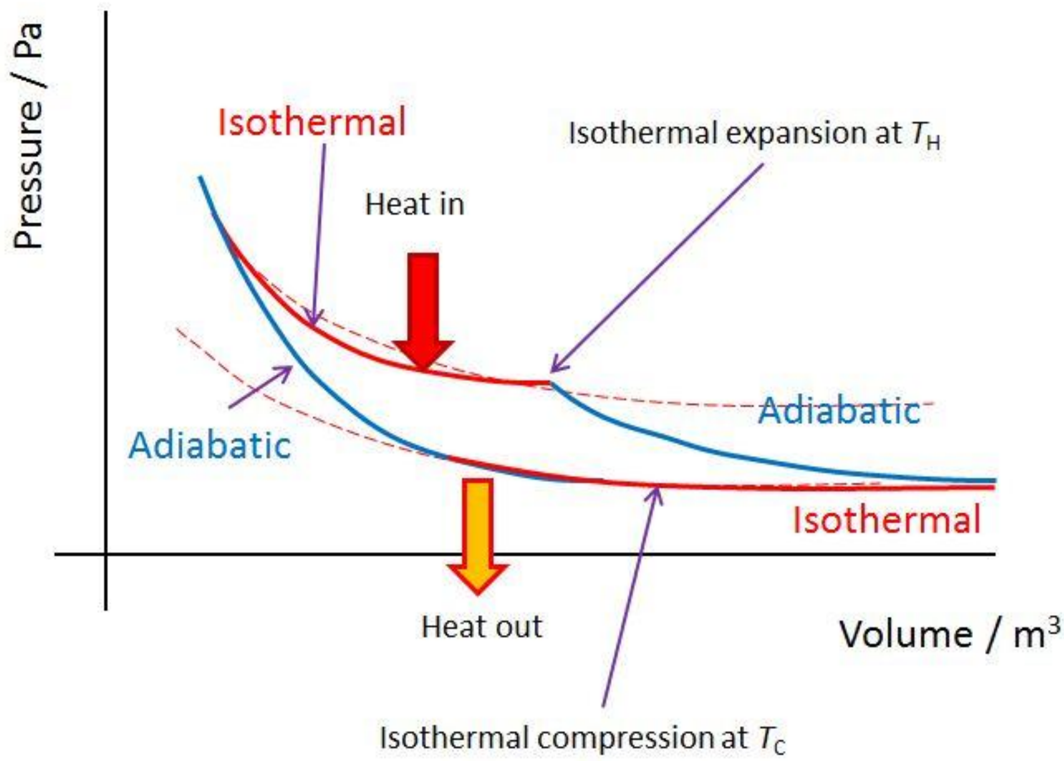


Figure 44 Carnot cycle

The Second Law of Thermodynamics tells us that not all the heat supplied to the system can be converted into mechanical work. The Carnot Efficiency tells us the maximum fraction of the supplied energy that can be converted into useful work.

The **Carnot Efficiency** is given by:

$$\text{Efficiency} = \frac{T_H - T_C}{T_H}$$

.... *Equation 74*

The picture (*Figure 45*) below shows the condenser of a steam turbine:

Figure 45 A power station condenser

In this case, steam enters the condenser at a temperature of 110 °C, and the cooling water temperature is about 15 °C.

The dipping duck in the photograph below is a heat engine (*Figure 46*):

Figure 46 A Dipping duck

There are limitations to the theoretical efficiency of any heat engine.

- T_H cannot be too high, otherwise components could melt.
- T_C will be in the normal range of atmospheric temperatures.
- Careful analysis of the cycle of an engine can help improve efficiency.
- Careful design of ports so that gas can get in and out with the minimum resistance.
- Friction cannot be eliminated. Lubrication reduces friction in bearings, but there is some viscous drag with the oils themselves.

A real engine does less work for a given heat transfer Q_{in}. Additionally, if we do a job of work on the real engine, we will not get Q_{in} back. The real engine is much less efficient than the reversible engine. If you are driving a car downhill in gear, the engine acts as a brake. It will not produce the same heat flow as it would if driving the car along a level road. Just as well, otherwise you would boil the engine going downhill. Not a good idea.

14C.063 Heat Pumps

We can reverse the process, pumping energy from the cold side to the hot side (*Figure 47*):

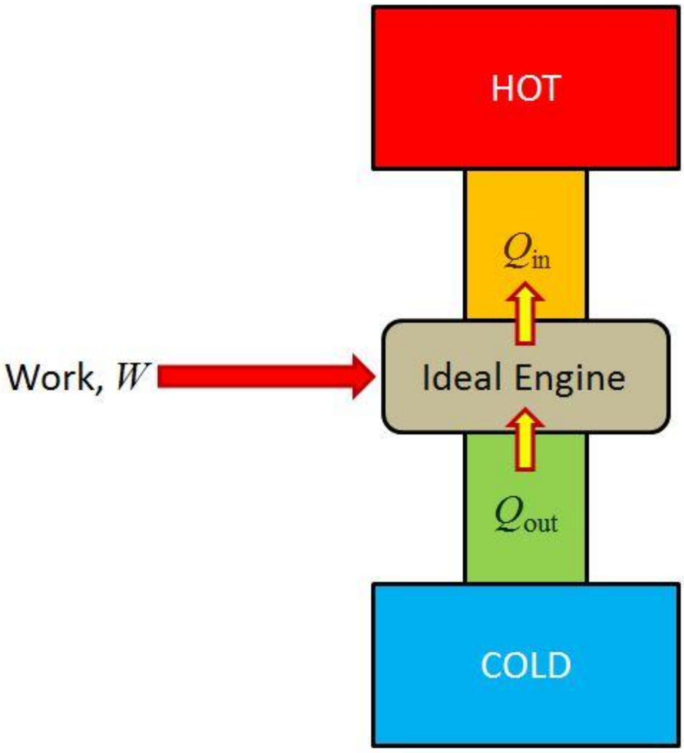


Figure 47 Pumping heat energy from cold to hot

Such a device is called a **heat pump**. The cold reservoir is the **environment**, which can be the ground, the air, or water. The hot reservoir is the house. In this case the heat pump is shifting the heat energy from the cold to the hot. The cold reservoir is very large; the hot reservoir is small. The idea is shown in the diagram (*Figure 48*).

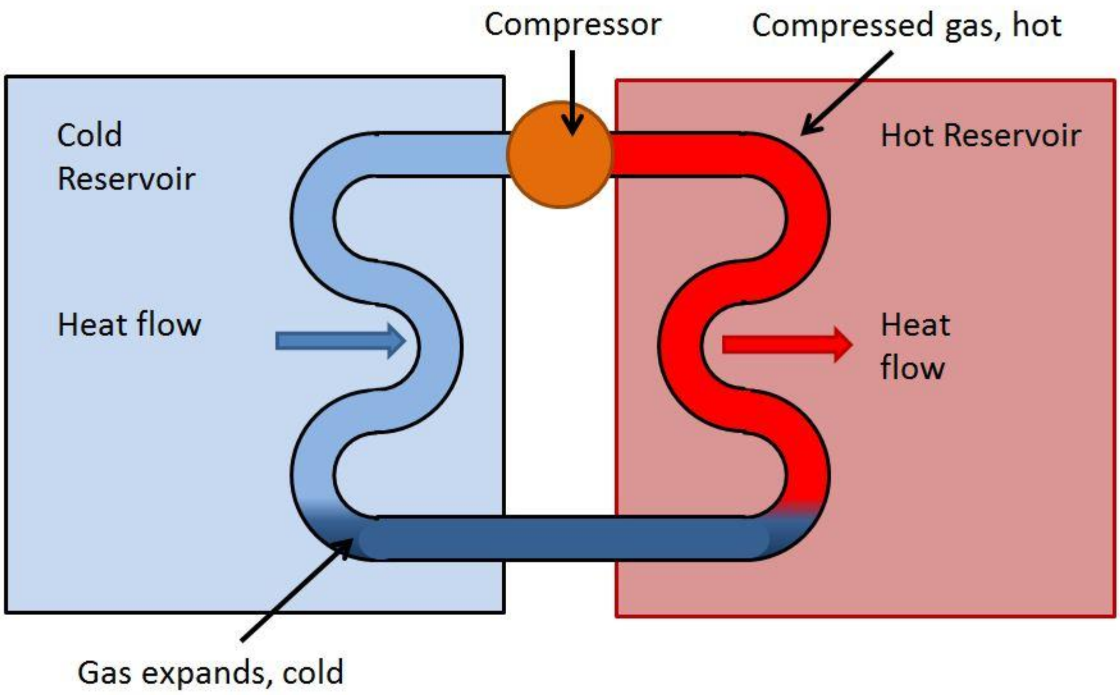


Figure 48 Generalised action of a heat pump

The pump moves the compressed gas which is hot. It releases the heat into the hot reservoir, say a room. The gas continues around the loop and expands cooling down. As it expands, heat flows from the cold reservoir, which may be the atmosphere, or the ground, or water. The gas is therefore warmed up by the heat flow into it. It is then compressed again, and the cycle continues.

Since the heat flow comes from the environment, which is the cold reservoir, a much greater amount of heat can be pumped into the room than the power of the compressor motor. The heat flow possible from the environment is almost limitless.

This is a good way of heating houses sustainably, but the equipment is expensive and planning permission may be needed.

An important quantity is the **coefficient of performance**. This is:

The ratio of the heat transferred to the hot reservoir to the input energy required to pump the heat from the cold to the hot.

For a heat pump, the coefficient of performance is given by:

$$CoP_{HP} = \frac{Q_H}{W} = \frac{Q_H}{Q_H - Q_C} = \frac{T_H}{T_H - T_C}$$

.......*Equation 75*

The coefficient of performance is a ratio, so it has no units. For a heat pump, the coefficient of performance can never be less than 1.

Worked example The output temperature of a heat pump needs to be 30 ° C, while the ground temperature is 0 °C. (a) Calculate the coefficient of performance. (b) The motor of the heat pump has a power of 1.5 kW. What is the power of such a heat pump heater?
Answer (a) Equation $CoP_{HP} = \frac{T_H}{T_H - T_C}$ Convert Celsius temperatures to Kelvin: 30 °C = 303 K 0 °C = 273 K CoP_{HP} = 303 K ÷ (303 K - 273 K) = **10.1** (b) Equation: $Q_H = CoP_{HP} \times W$ Q_H = 10.1 × 1.5 kW = 15.15 kW = **15 kW** (2 s.f.)

In reality, the coefficient of performance of such a situation is rather less than the theoretical maximum.

The picture (*Figure 49*) below shows a heat pump:

Figure 49 A domestic heat pump (By Kristoferb at English Wikipedia, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=10795550)

The heat pump provides about 3 to 4 times as heat than a simple resistance heater. However, the installations are time consuming, very disruptive and rather expensive. Additionally, there are strict building regulations as to where these devices should be situated. The pump needs to be away from the house and not within 1.5 metres of a boundary. Many houses do not have a sufficient plot size to accommodate such a machine. The effectiveness of heat pumps as a primary source of heating has been questioned. Using at least 10 kW of electrical power, they will rapidly run up large electricity bills.

In a **refrigerator**, the cold reservoir is small, and heat is pumped from the cold reservoir to the hot, which is the environment (i.e. the kitchen where the fridge stands). While this may appear to contravene the Second Law of Thermodynamics, remember that the source of energy is the real hot reservoir which is the boiler in the power station that produces the energy to turn the turbine to generate the electricity.

The compressor motor pumps and compresses a **coolant**. The compressed coolant goes to a **heat exchanger** outside on the back, where heat is transferred by **convection** into the room. The coolant then is sprayed into an **expansion chamber** in the ice compartment of the fridge. The liquid **evaporates**, which requires energy as **latent heat**. The energy required is taken from the inside of the cabinet, which is heavily insulated to prevent heat flowing from the room. The coolant gas is taken back to the pump.

For a refrigerator, the coefficient of performance is given by:

$$CoP_{\text{ref}} = \frac{Q_{\text{C}}}{W} = \frac{Q_{\text{C}}}{Q_{\text{H}} - Q_{\text{C}}} = \frac{T_{\text{C}}}{T_{\text{H}} - T_{\text{C}}}$$

.... *Equation 76*

A way of telling which of the coefficient of performance formulae to use is to consider the target reservoir. If it's a room being heated, we use the CoP_{HP} formula. If it's space that's being cooled, then we use the CoP_{ref} formula.

We can work out the theoretical coefficient of performance for a refrigerator.

Worked example A refrigerator can maintain the contents at a temperature of 4 °C while the room has a temperature of 30 °C. (a) Calculate the coefficient of performance. (b) The motor has a power of 250 W. What is the rate at which heat is transferred from the cabinet to the room?
Answer (a) Use: $$CoP_{\text{ref}} = \frac{T_{\text{C}}}{T_{\text{H}} - T_{\text{C}}}$$ Convert the Celsius temperatures to Kelvin: 4 °C = 277 K. 30 °C = 303 K CoP_{ref} = 277 K ÷ (303 K - 277 K) = **10.7** (b) $Q_{\text{C}} = CoP_{\text{ref}} \times W$ = 10.7 × 250 W = 2663 W = **2700 W** (2 s.f.)

14C.064 Other Laws of Thermodynamics (Extension)

Thermal equilibrium happens when two bodies are brought into contact with each other, there is no heat flow. This means that the temperature of the body remains the same. Consider the three bodies, A, B, and C below (*Figure 50*). The brown arrows show the heat flows between the three bodies.

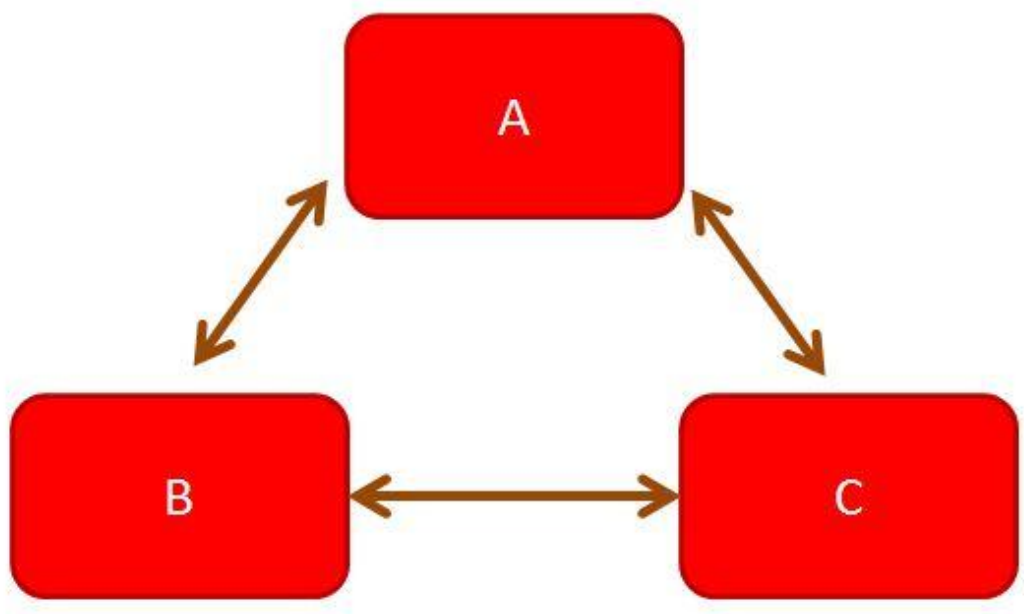


Figure 50 Heat flows between three bodies in thermal equilibrium

According to the **Zeroth Law of Thermodynamics**, if A and B are in thermal equilibrium, and A and C are in thermal equilibrium, then B and C must be in thermal equilibrium.

The law was called this because the First and Second laws of Thermodynamics depend on it. It was discovered later than the First and Second Laws.

The **Third Law of Thermodynamics** states that:

The entropy of a perfect crystal is exactly zero when the temperature is absolute zero.

Questions

Tutorial 14C.06

14C.06.1

A car uses energy from the fuel at a rate of 72 kJ s^{-1}. It uses 9 kJ s^{-1} to move along the road. How much heat is lost as waste? What is the efficiency?

14C.06.2

What is the maximum possible efficiency of an engine using steam at a temperature of 100 °C on a day when the temperature is 24 °C?

14C.06.3

Look at *Figure 43*. What do you think are the hot and cold reservoirs?

14C.06.4

A small geothermal power station in Iceland pumps cold water into hot rock strata far below the Earth's surface to be heated and returned at a constant temperature of 87 °C. The power station uses the hot water as the heat source for a heat engine which rejects energy to the much colder sea water near the station.

(a) When the temperature of the sea water is 7 °C the power output from the heat engine is 5.0MW. Calculate:

(i) the maximum theoretical efficiency of the heat engine,

(ii) the rate at which heat energy must be transferred from the hot water if the engine works at the maximum theoretical efficiency,

(iii) the rate at which energy must be transferred to the sea water under these conditions.

(b) The power station produces electrical power with an overall efficiency which is much lower than the maximum theoretical efficiency of the heat engine. Give two reasons for this lower efficiency.

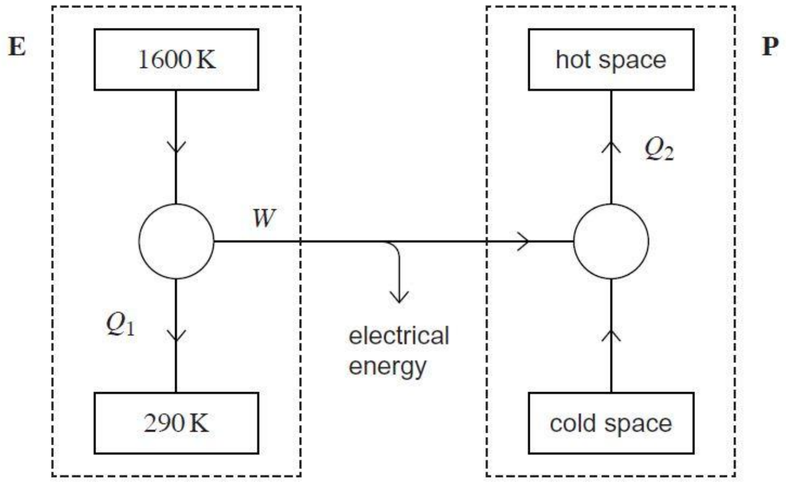
(c) The overall efficiency of an oil-fired power plant of similar size to the geothermal station is over four times as great. Suggest one reason, other than less pollution, why the geothermal source was preferred for the power station.

(AQA Past question)

14C.06.5

(a) Explain what is meant by the coefficient of performance of a heat pump.

(b) The box labelled **E** in Figure 3 shows a diagram of a combined heat and power scheme. The scheme provides electrical energy *W* from an engine-driven generator and heat Q_1 for buildings situated near to the generator. Some of the electrical energy is used to drive the heat pump shown in the box labelled **P**. Output Q_2 is also used to heat the buildings.



You may assume that the engine runs at its maximum theoretical efficiency and that the electrical generator is 100% efficient. The output power of the engine-driven generator is 80kW.

(i) The fuel used in the engine (**E**) is propane of calorific value 49 MJ kg^{-1}. Calculate the rate of flow of propane into the engine. State an appropriate unit.

(ii) The heat pump has a coefficient of performance of 2.6. The power supplied by the electrical generator to the heat pump (**P**) is 16 kW. Calculate the total rate at which energy is available for heating from both the engine and heat pump.

(iii) The conversion of electrical energy to heat is nearly 100% efficient. Explain why the designer has proposed installing a heat pump rather than an electrical heater to provide the additional heat Q_2

(AQA past question June 2014 Q4)

Tutorial 14C.07 Deriving the Moment of Inertia Equations	
SQA Advanced Higher (14C.075), Cambridge Pre-U and Extension only	
Contents	
14C.071 Summing Moments of Inertia	14C.072 Moments of Inertia by Calculus
14C.073 Moment of Inertia for a Disc	14C.074 Moment of Inertia for a Ring
14C.075 The Moment of Inertia for a Rod. (SQA Advanced Higher)	14C.076 Three Cases

You are NOT expected to know this for the AQA Engineering Option. This is an EXTENSION only. You may come across it in university level physics.

This tutorial is long and is quite challenging. It will only be assessed in Paper 3 Section 2 (the hard bit).

I would advise you to do each part separately and work through it quite slowly.

Before doing so, have a good cup of coffee.

We have seen how any object that has a mass has **inertia**. See Topics 5 and 8. In linear dynamics, we saw that inertia is the amount by which an object resists change in motion. We know from Newton II that the change in motion is described by the acceleration. We have also seen in the previous tutorial that there is an equivalent quantity in rotational motion called **moment of inertia**. The moment of inertia is the amount by which a rotating object opposes angular acceleration, and is related to the mass m by the equation:

$$I = \Sigma mr^2$$

.... *Equation 77*

14C.071 Summing Moments of Inertia

We can add the moments of inertia for individual elements simply by adding them up. Suppose we had three masses m_1, m_2, and m_3 at radius r_1, r_2, and r_3 respectively. Each is held to the axis by a very light stiff rod of negligible mass. They are rotating anticlockwise around a vertical axis (strictly speaking, the ***z*-axis**). See *Figure 51*.

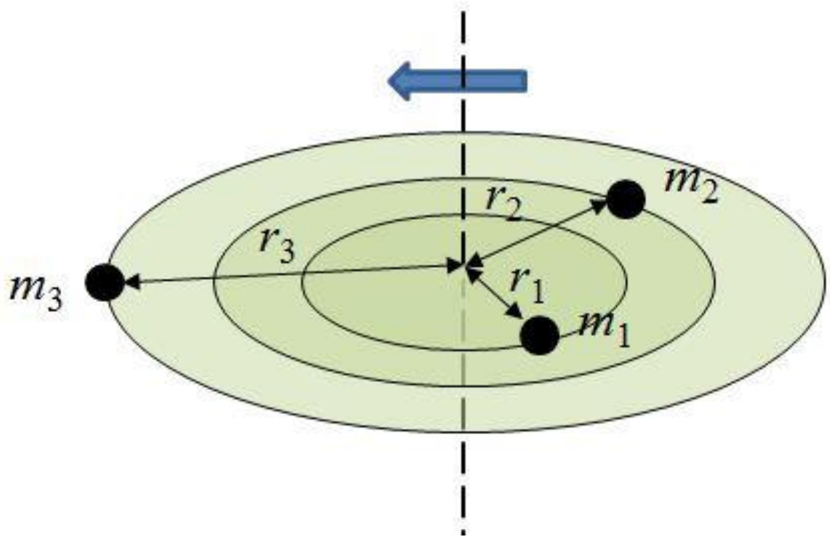


Figure 51 Summing moments of inertia

Therefore, for each element, we can write the moment of inertia for each one:

$$I_1 = m_1 {r_1}^2$$

$$I_2 = m_2 {r_2}^2$$

$$I_3 = m_3 {r_3}^2$$

.... *Equation 78*

The total moment of inertia is the sum of the three individual moments of inertia:

$$I_T = I_1 + I_2 + I_3$$

.... *Equation 79*

So, we can write:

$$I_T = m_1 r_1^{\,2} + m_2 r_2^{\,2} + m_3 r_3^{\,2}$$

.... *Equation 80*

Worked example Three masses of 2.0 kg, 2.5 kg, and 3.0 kg are at radii 1.0 m, 1.5 m, and 2.0 m respectively. Calculate the total moment of inertia.
Answer Calculate the moment of inertia for each element: I_1 = 2.0 kg × (1.0 m)^2 = 2.0 kg m^2. I_2 = 2.5 kg × (1.5 m)^2 = 5.625 kg m^2. I_3 = 3.0 kg × (2.0 m)^2 = 12 kg m^2. Now add these together: I_T = 2.0 kg m^2 + 5.625 kg m^2 + 12 kg m^2 = 19.625 kg m^2 = **20 kg m^2** (to 2 s.f. as data are to 2 s.f.)

Notice that the moment of inertia is **independent** of the angular velocity. Nor have we considered whether the system will balance. You could work out whether it would balance by working out the centripetal forces from each element and then using the principle of 3 co-planar forces. If the three centripetal forces sum to zero, the system will balance. Unbalanced rotating systems end up going all over the place. Not a good idea.

Note that the moment of inertia is a **scalar** quantity.

14C.072 Deriving Moments of Inertia by Calculus

In the section above, we gave a numerical value to each mass element but treated them as point masses. In reality, they would have a certain volume. Therefore, each of the three elements can be described as a **volume element**. Each element will have a value of **density** depending on the material it's made of. In these derivations, we will assume that all systems have a constant density. We can define the moment of inertia dI for a small volume element dV of small mass dm as:

$$dI = r^2 dm$$

.... *Equation 81*

The r term is the radius perpendicular to the central axis or z-axis. We will consider only the z-axis.

To get the total moment of inertia, we need to add up all the moments of inertia of all the volume elements. This is denoted by the Sigma (Σ):

$$I_T = \sum dI = \sum r^2 dm$$

.... *Equation 82*

It would seem an obvious step to integrate this between radius r_1 and radius r_2:

$$I_T = \int_{r_1}^{r_2} r^2 dm$$

.... *Equation 83*

Using the powers rule, we get:

$$I_T = m\left[\left(\frac{r_2{}^3}{3}\right) - \left(\frac{r_1{}^3}{3}\right)\right]$$

.... *Equation 84*

This, however, seems to be quite unlike any of the relationships we saw in the previous page. We have to do more.

14C.073 Moment of Inertia for a Disc

Consider a thin rotating disk of radius r and thickness t, where the thickness is very much less than the radius ($t << r$). The disk is rotating around the z axis. The disk also has a uniform density of ρ. It is shown below (*Figure 52*):

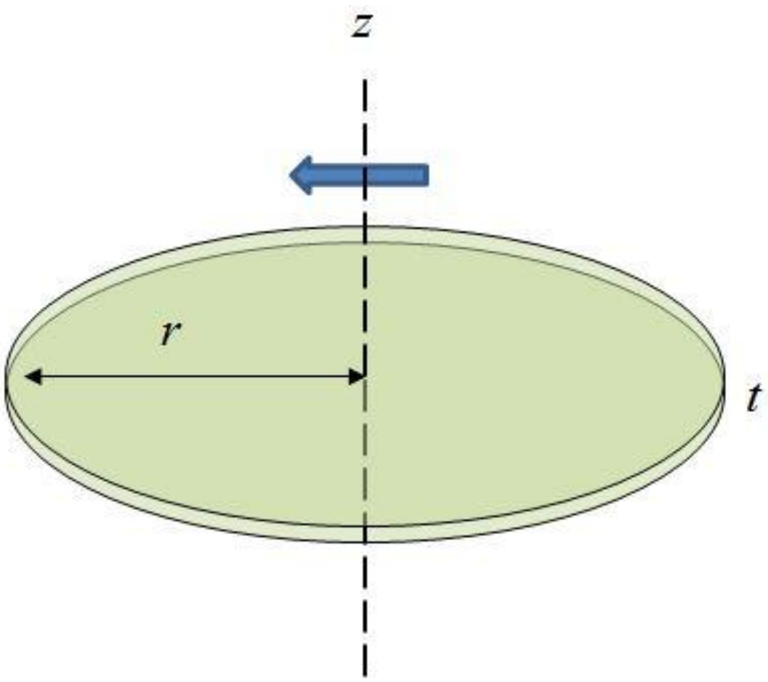


Figure 52 Moment of inertia for a disc

To get the total moment of inertia, I_T, we need to sum all the moments of inertia from the centre to the radius, r. This is summed up by the integration:

$$I_T = \int_0^r r^2 dm$$

.... *Equation 85*

The next step is not immediately intuitive, but we know that:

mass = volume × density

$$m = V\rho$$ .... *Equation 86*

This links in with the volume elements mentioned above in *Equation 86*. So, we can rewrite this as:

$$dm = \rho dV$$ .... *Equation 87*

We can say that the disk is made up of a number of very narrow rings arranged concentrically. We can now consider the volume of each small element as a very narrow ring of **rectangular** cross-section. The ring is so narrow that the outer radius is only negligibly larger than the inner radius. Therefore, the cross sectional area is given by the width of the ring multiplied by the thickness:

$$A = dr \times t$$ .... *Equation 88*

The small volume of the very narrow ring can be worked out by multiplying the area by the circumference of the ring:

$$dV = 2\pi r \times dr \times t$$ .... *Equation 89*

So, we can now work out the mass of each very narrow ring:

$$dm = \rho(2\pi r dr)t$$ .... *Equation 90*

Now we can go back to the integral equation:

$$I_T = \int_0^r r^2 dm$$ .... *Equation 91*

And we substitute for dm to give:

$$I_T = 2\pi\rho t \int_0^r r^3 \, dr$$

.... *Equation 92*

So, we use the powers rule and take the constant of integration as 0. Therefore:

$$I_T = 2\pi\rho t \left[\frac{r^4}{4} - 0\right] = \frac{1}{2}\pi\rho t r^4$$

.... *Equation 93*

We are not quite there yet... We need to consider the total mass of the disk, which we will call *M*. We know that:

$$M = V\rho$$ .... *Equation 94*

The total volume of the disk is its area × thickness:

$$V = \pi r^2 t$$ .... *Equation 95*

So, we can now write:

$$M = \rho\pi r^2 t$$ .... *Equation 96*

Rearranging:

$$\frac{M}{r^2} = \pi\rho t$$

.... *Equation 97*

Now our last step is to substitute for $\pi\rho t$:

$$I_T = \frac{M}{2r^2} r^4 = \frac{1}{2} M r^2$$

.... *Equation 98*

We saw this result in the previous tutorial (14C.01) for a rotating disc and a rotating cylinder. It true for a rotating cylinder because a cylinder is simply a disc with a large value for thickness.

Worked example A solid disc has a radius of 20 cm and a mass of 1.2 kg. Calculate the moment of inertia.
Answer I = 1/2 × 1.2 kg × (0.20 m)^2 = **0.024 kg m^{-2}**.

14C.074 Moment of Inertia for a Ring

Consider a thin rotating ring (often called an **annulus**) of inner radius r_1, outer radius r_2, and thickness t, where the thickness is very much less than the radius ($t << r$). The ring is connected to the axis by very fine spokes of negligible mass. The disk is rotating around the z axis. The disk also has a uniform density of r. It is shown below (*Figure 53*):

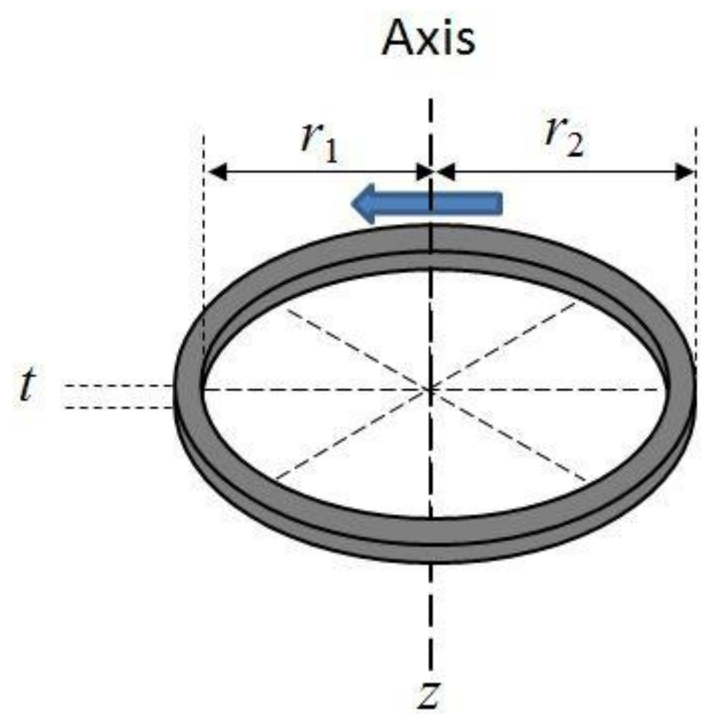


Figure 53 Moment of inertia for a ring

As before, we can define the moment of inertia dI for a small volume element dV of small mass dm as:

$$dI = r^2 dm$$ .... *Equation 99*

The r term is the radius perpendicular to the central axis or z-axis. We will consider only the z-axis.

For the ring, the integration is different to the disk in that we have to take into account the inner radius and the outer radius. Therefore, the integral equation is:

$$I_T = \int_{r_1}^{r_2} r^2 dm$$ .... *Equation 100*

We will assume that the ring has a uniform density. Therefore, we can work out the mass, dm, of each small volume element:

$$dm = \rho dV$$ .... *Equation 101*

As before, we can say that the volume elements are rings of very narrow width, dr, so that the difference between the outer radius and the inner radius is so small as to be negligible. As before the rings are of **rectangular** cross-section. Therefore, the cross sectional area is given by the width of the ring multiplied by the thickness:

$$A = dr \times t$$ .... *Equation 102*

The small volume of the very narrow ring can be worked out by multiplying the area by the circumference of the ring:

$$dV = 2\pi r \times dr \times t$$ *............... Equation 103*

So, we can now work out the mass of each very narrow ring:

$$dm = \rho(2\pi r dr)t$$ *.............Equation 104*

So, we can write the moment of inertia:

$$I_T = \int_{r_1}^{r_2} r^2 dm = 2\pi\rho t \left(\int_{r_1}^{r_2} r^3 dr \right)$$

.... Equation 105

Integrating between r_2 and r_1, this works out as:

$$I_T = 2\pi\rho t \left(\frac{r_2{}^4 - r_1{}^4}{4} \right) = \frac{1}{2}\pi\rho t(r_2{}^4 - r_1{}^4)$$

.......Equation 106

We can factorise this as:

$$I_T = \frac{1}{2}\pi\rho t(r_2{}^2 - r_1{}^2)(r_2{}^2 + r_1{}^2)$$

.... Equation 107

The total mass M of the ring is the difference between the mass of the disk of radius r_2 and the mass of a disk of radius r_1:

$$M = M_2 - M_1 = \pi\rho t r_2{}^2 - \pi\rho t r_1{}^2 = \pi\rho t(r_2{}^2 - r_1{}^2)$$ *......... Equation 108*

We can rearrange this to:

$$(r_2{}^2 - r_1{}^2) = \frac{M}{\pi\rho t}$$

.... *Equation 109*

and then substitute into *Equation 107*:

$$I_\mathrm{T} = \frac{1}{2}\pi\rho t(r_2{}^2 - r_1{}^2)(r_2{}^2 + r_1{}^2)$$

which gives:

$$I_\mathrm{T} = \frac{1}{2}\pi\rho t\frac{M}{\pi\rho t}(r_2{}^2 + r_1{}^2)$$

.... *Equation 110*

Cancelling gives us our final result:

$$I_\mathrm{T} = \frac{1}{2}M(r_2{}^2 + r_1{}^2)$$

.... *Equation 111*

Worked example A solid disc has a outer radius of 20 cm, an inner radius of 17 cm, and a mass of 1.2 kg. Calculate the moment of inertia.
Answer I = 1/2 × 1.2 kg × [(0.20 m)^2 + (0.17 m)^2] = **0.041 kg m^{-2}**

This result tells us that a ring of the same mass as the disk has a greater moment of inertia. Let's suppose that the inner radius is $0.85\ r_2$ where r_2 is the outer radius. The thickness of the ring is the same as the disk, t. Substituting gives us:

$$I_\mathrm{T} = \frac{1}{2}M\left(r_2{}^2 + 0.85^2 r_2{}^2\right)$$

.... *Equation 112*

Therefore:

$$I_T = \frac{1}{2}M{r_2}^2(1 + 0.85^2) = 0.86M{r_2}^2$$

.... *Equation 113*

The moment of inertia for a ring is therefore 0.86 ÷ 0.50 = 1.72 times the moment of inertia of a disk. Using the answers to the two worked examples, we divide the moment of inertia of the ring by the moment of inertia of the disk. We find that, to two significant figures, the answers are consistent.

If the inner radius and the outer radius are (nearly) the same:

$$I_T = Mr^2$$

.... *Equation 114*

Clearly this would be an impossible situation, but what we can say is that the moment of inertia for a ring lies in the range of between 1 and 2 times the moment of inertia of a disk. To achieve the same mass as a disk, however, we could have a material of higher density, as the volume of the material will be less. If the ring is made of the same material and thickness as the disc, the mass will be less. We can achieve the same mass of material of the same density by having an increase in thickness around the outside. In engineering, practical flywheels are made in this way (*Figure 54*).

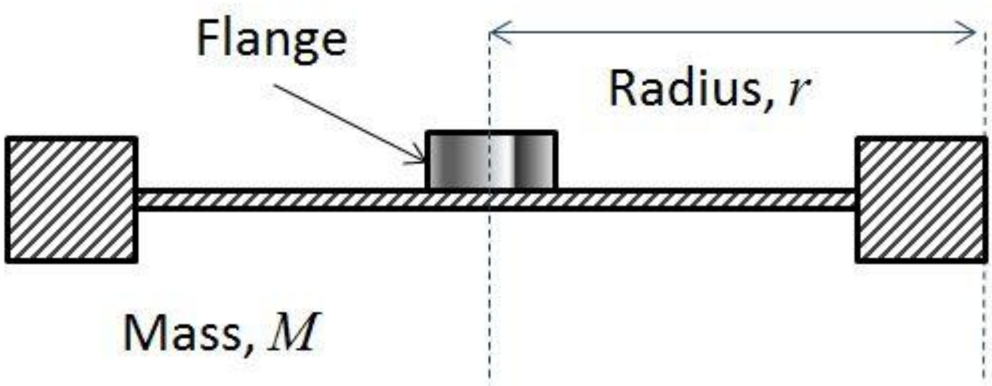


Figure 54 A practical flywheel

The picture below (*Figure 55*) shows the platter of a vinyl LP record deck (upside down). Note how there is a thick rim to increase the moment of inertia (hence the angular momentum):

Figure 55 The turntable (upside down) of a record deck

14C.075 The Moment of Inertia for a Rod (SQA Advanced Higher)

We will now derive equations for a rod rotating about an axis that is perpendicular to its length. The simplest case is where the axis passes through the centre of the rod. We will assume that the rod is uniform. The rod has mass M and length L. The rod has a **mass per unit length**, μ, which is given by:

$$\mu = \frac{M}{L}$$

The term μ has nothing to do with coefficient of friction.

14C.076 Three cases

Case 1 Rod rotating about a central axis

The rod is rotated about the Z-axis like this like this (*Figure 56*):

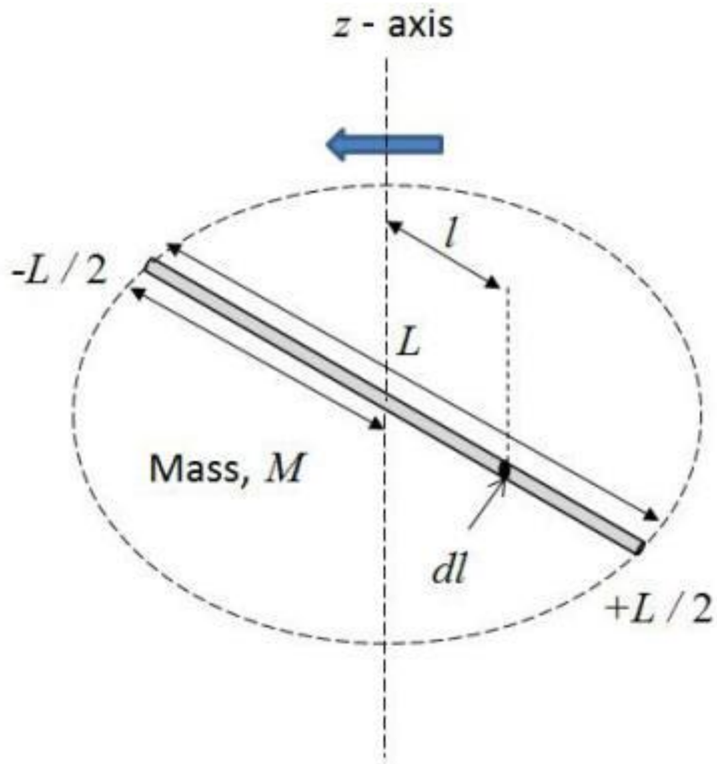


Figure 56 Road being rotated around its centre.

The distance from the z-axis to each end of the rod is $L/2$. We have used the **axis** as the **zero** point. Anything that is to the left we will say is **negative**. Anything to the right is **positive**.

We know the definition of the moment of inertia is given by:

$$I_{\mathrm{T}} = \int_{r_1}^{r_2} r^2 dm$$

.... *Equation 115*

We will consider very small element of length dl that is l from the z-axis. The element has a very small mass dm.

Since the rod is uniform, we can say that mass per unit length, μ is:

$$\mu = \frac{M}{L} = \frac{dm}{dl}$$ .... *Equation 116*

We can rearrange this to give:

$$dm = \frac{M}{L} dl$$ .... *Equation 117*

We can modify our general equation by writing

- $r = l$
- $r_1 = -L/2$
- $r_2 = +L/2$

The terms M and L are constant. So, the equation becomes:

$$I = \frac{M}{L} \int_{-L/2}^{+L/2} l^2 \, dl$$ .... *Equation 118*

The result of this integration is:

$$I = \frac{M}{L}\left[\left(\frac{\left(+L/2\right)^3}{3}\right) - \left(\frac{\left(-L/2\right)^3}{3}\right)\right]$$ .... *Equation 119*

This simplifies to:

$$I = \frac{M}{3L}\left[\left(\frac{+L}{2}\right)^3 - \left(\frac{-L}{2}\right)^3\right]$$ .... *Equation 120*

And further to:

$$I = \frac{M}{3L}\left(\frac{L^3}{8} + \frac{L^3}{8}\right)$$

.... *Equation 121*

To give:

$$I = \frac{M}{3L} \times \frac{2L^3}{8}$$

.... *Equation 122*

Cancelling:

$$I = \frac{M}{3} \times \frac{L^2}{4}$$

.... *Equation 123*

And we have our final result:

$$I = \frac{ML^2}{12}$$

.... *Equation 124*

We've got there in the end.

Examples of this case might include simplified models of the rotor of a helicopter, the blade of a rotary lawnmower, or a two bladed propellor of a light aeroplane.

Case 2 Rod rotating about an axis at the end

We will use the same rod as we did before, except that the z-axis is at one end. We will assume that the rod is uniform. The rod has mass M and length L. The rod has a **mass per unit length**, μ, which is given by:

$$\mu = \frac{M}{L}$$ .... *Equation 125*

The rod rotates about the axis like this (*Figure 57*):

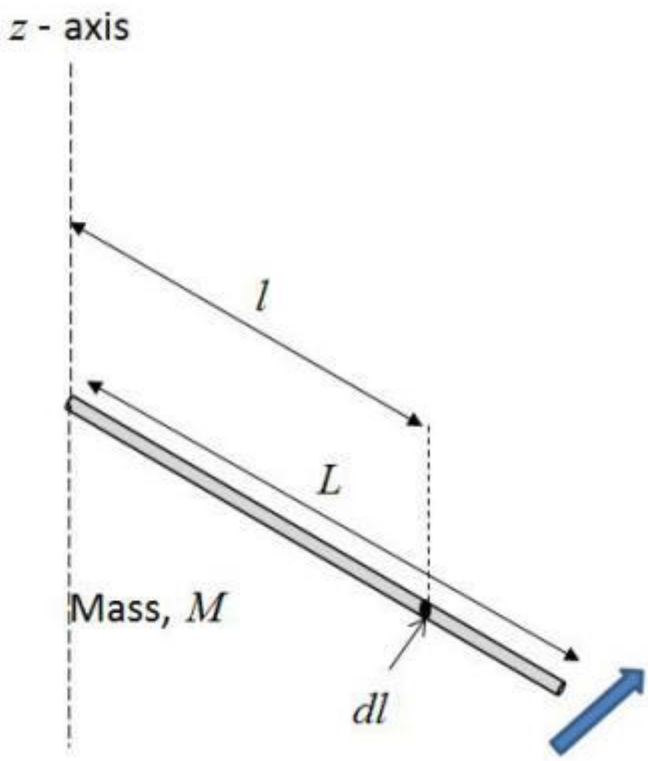


Figure 57 Rod spinning about an axis at one end

We have used the **axis** as the **zero** point. The other end of the rod is a distance +*L* from the axis. We know the definition of the moment of inertia is given by:

$$I_{\mathrm{T}} = \int_{r_1}^{r_2} r^2 dm$$ .... *Equation 126*

We will consider very small element of length dl that is l from the z-axis. The element has a very small mass dm. Since the rod is uniform, we can say that:

$$\mu = \frac{M}{L} = \frac{dm}{dl}$$ .... *Equation 127*

We can rearrange this to give:

$$dm = \frac{M}{L}dl$$ .... *Equation 128*

We can modify our general equation by writing

- $r = l$
- $r_1 = 0$
- $r_2 = +L$

The terms M and L are constant. So, the equation becomes:

$$I = \frac{M}{L}\int_0^L l^2\,dl$$ .... *Equation 129*

The result of the integration is:

$$I = \frac{M}{L} \times \frac{L^3}{3}$$ .... *Equation 130*

This tidies up to:

$$I = \frac{ML^2}{3}$$ .... *Equation 131*

Which is our final result.

Case 3 Rod rotating about an axis that is neither at the centre nor one end

We will use the same rod as we did before, except that the z-axis is at a point h from the left hand end. We will assume that the rod is uniform. The rod has mass M and length L. The rod has a **mass per unit length**, μ, which is given by:

$$\mu = \frac{M}{L}$$ .... *Equation 132*

The rod rotates about the axis like this (*Figure 58*):

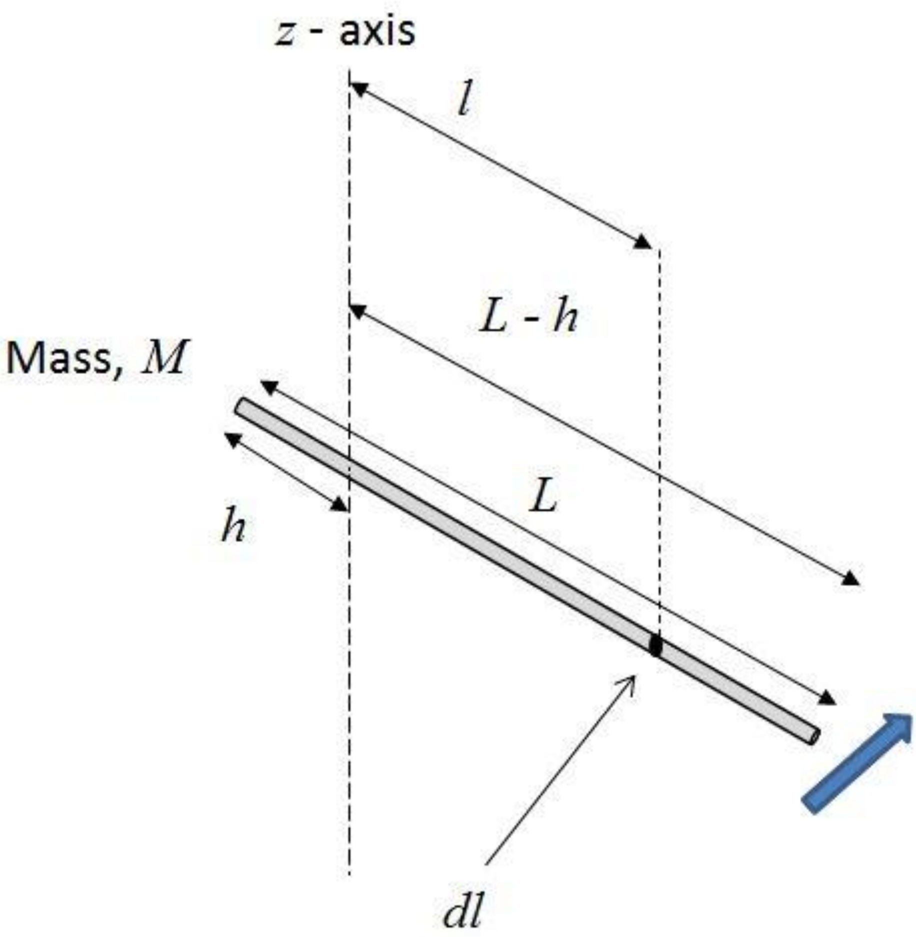


Figure 58 Rod of unequal lengths rotating about an axis

The **axis** is the **zero** point. The left hand end is $-h$ from the axis. The right hand end of the rod is a distance $+(L - h)$ from the axis. We know the definition of the moment of inertia is given by:

$$I_{\mathrm{T}} = \int_{r_1}^{r_2} r^2 dm$$

.... *Equation 133*

We will consider very small element of length dl that is l from the z-axis. The element has a very small mass dm. Since the rod is uniform, we can say that:

$$\mu = \frac{M}{L} = \frac{dm}{dl}$$

.... *Equation 134*

We can rearrange this to give:

$$dm = \frac{M}{L} dl$$

.... *Equation 135*

We can modify our general equation by writing

- $r = l$
- $r_1 = -h$
- $r_2 = +(L - h)$.

The terms M and L are constant. So, the equation becomes:

$$I = \frac{M}{L} \int_{-h}^{L-h} l^2 dl$$

.... *Equation 136*

This results in:

$$I = \frac{M}{L} \left[\frac{(L - h)^3}{3} - \frac{-h^3}{3} \right]$$

.... *Equation 137*

This can be simplified to:

$$I = \frac{M}{3L}[(L-h)^3 + h^3]$$

.... Equation 138

We need to expand the $(L - h)^3$ term:

$$(L-h)^3 = L^3 - 3L^2h + 3Lh^2 - h^3$$

.... Equation 139

So, we put this in to our equation:

$$I = \frac{M}{3L}[L^3 - 3L^2h + 3Lh^2 - h^3 + h^3]$$

.... Equation 140

This simplifies to our final result:

$$I = \frac{1}{3}M(L^2 - 3Lh + 3h^2)$$

.... Equation 141

If you have got this far, well done

Now have another coffee.

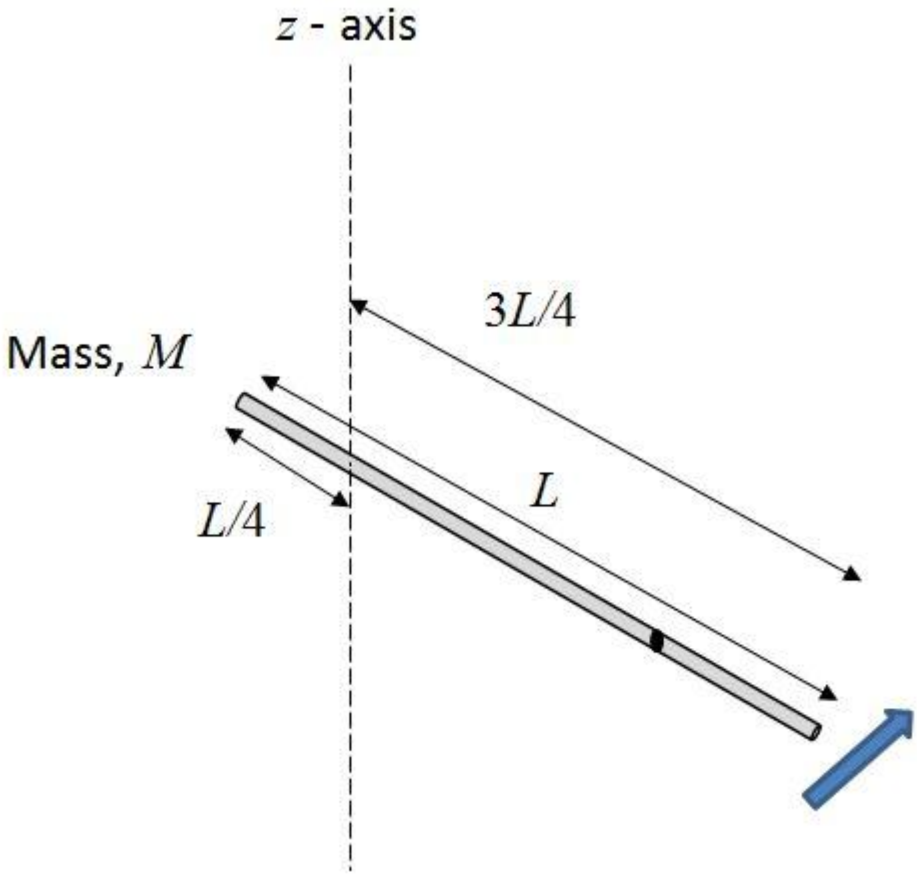
Don't do any more until tomorrow.

Questions

Tutorial 14C.07

14C.07.1 (Challenge)

A rod has a mass of M and a length L. It spins on an axis that is perpendicular to the rod and passes through the rod at a point $L/4$ of its length. This is shown below:



(a) Using calculus techniques, show that the moment of inertia of the rod is given by:

$$I = \frac{7ML^2}{48}$$

(b) The rod is 1.2 m long and is of circular cross-section 1.0 cm in diameter. The rod is made of steel, density 7600 kg m^{-3}. Calculate its mass.

(c) Hence calculate the moment of inertia. State the unit.

Answers to Questions

Tutorial 14C.01

14C.01.1

Use Angular velocity = 2π(revs min^{-1} ÷ 60 s min^{-1})

$\omega = 2 \times \pi \times (33\ min^{-1} \div 60\ s\ min^{-1}) = 2 \times \pi \times 0.55\ s^{-1} =$ **3.46 rad s^{-1}**

14C.01.2

Angular acceleration is in rad s^{-2}

(Just like linear acceleration is in m s^{-2})

14C.01.3

Acceleration = change in angular velocity ÷ time interval

Angular velocity = $2 \times \pi \times (3000\ min^{-1} \div 60\ s\ min^{-1}) = 100\pi$ rad s^{-1}

$\alpha = (100\pi - 0)$ rad $s^{-1} \div 5$ s $= 20\pi$ rad s^{-1} = **62.8 rad s^{-2}**

14C.01.4

For the circular disc

$$I = \frac{Mr^2}{2}$$

I = [2.5 kg x $(0.2\ m)^2$] ÷ 2 = 0.05 kg m^2

For the solid sphere:

$$I = \frac{2Mr^2}{5}$$

I = [2 x 2.5 kg x $(0.2\ m)^2$] ÷ 5 = **0.04 kg m^2**

14C.01.5

(a) Flywheel is a hollow cylinder:

$I = Mr^2$ = 160 kg x (0.34 m)^2 = **18.5 kg m^2**

(b) Find the angular velocity first:

ω = 2 × π × (44000 min^{-1} ÷ 60 s min^{-1}) = 4608 rad s^{-1}

$$E_k = \frac{1}{2} I \omega^2$$

E_k = 1/2 × 18.5 kg m^2 × (4608 rad s^{-1})^2 = **1.96 × 10^8 J**

14C.01.6

The key to this is the moment of inertia. The moment of inertia of the solid disk flywheel (ignoring the flange in the middle) is given by:

$$I = 1/2\ Mr^2$$

The second flywheel can, to a first approximation, be considered as a cylinder of mean radius, r. The assumptions are

- that the thin part of the wheel has a mass that is very much smaller that the thick ring part.
- the flange can be ignored.

The moment of inertia is given by:

$$I = Mr^2$$

Therefore, the second flywheel will have a moment of inertia that is double that of the first. Therefore, it will store double the kinetic energy for any given angular velocity.

14C.01.7

Find the angular velocity at the start:

$$\omega_1 = 2 \times \pi \times (750\ \text{min}^{-1} \div 60\ \text{s min}^{-1}) = 25\pi = 78.5\ \text{rad s}^{-1}$$

Find the angular velocity and the end:

$$\omega_2 = 2 \times \pi \times (1500\ \text{min}^{-1} \div 60\ \text{s min}^{-1}) = 50\pi = 157\ \text{rad s}^{-1}$$

Now use the formula:

$$\omega_2 = \omega_1 + \alpha t$$

$$157\ \text{rad s}^{-1} = 78.5\ \text{rad s}^{-1} + \alpha \times 3\ \text{s}$$

Rearrange:

$$\alpha = 78.5\ \text{rad s}^{-1} \div 3\ \text{s} = \underline{\mathbf{26.2\ rad\ s^{-2}}}$$

14C.01.8

Find the angular velocity at the start:

$$\omega_1 = 2 \times \pi \times (750\ \text{min}^{-1} \div 60\ \text{s min}^{-1}) = 25\pi = 78.5\ \text{rad s}^{-1}$$

$$\text{Angular acceleration } \alpha = 78.5\ \text{rad s}^{-1} \div 3\ \text{s} = 26.2\ \text{rad s}^{-2}$$

Now use the formula:

$$\theta = \omega_1 t + 1/2 \alpha t^2$$

$$\theta = 78.5\ \text{rad s}^{-1} \times 3\ \text{s} + 1/2 \times 26.2\ \text{rad s}^2 \text{ x } (3\ \text{s})^2$$

$$\theta = 235.5\ \text{rad} + 117.9\ \text{rad} = 353.4\ \text{rad}$$

14C.01.9

Find the angular velocity at the start:

$$\omega_1 = 2 \times \pi \times (30000 \text{ min}^{-1} \div 60 \text{ s min}^{-1}) = 1000\pi = 3142 \text{ rad s}^{-1}$$

Angular displacement

$$\theta = 2/3 \times \pi = 2.09 \text{ rad}$$

Now use the formula:

$$0 = \omega_1^2 + 2\alpha\theta$$

$$0 = (3142 \text{ rad s}^{-1})^2 + 2 \times \alpha \times 2.09 \text{ rad}$$

$$\alpha = 9.87 \times 10^6 \text{ rad}^2 \text{ s}^{-2} \div -(2 \times 2.09 \text{ rad}) = \mathbf{-2.36 \times 10^6 \text{ rad s}^{-2}}$$

Negative means that it's slowing down.

14C.01.10

Find the angular velocity at the start:

$$\omega_1 = 2 \times \pi \times (750 \text{ min}^{-1} \div 60 \text{ s min}^{-1}) = 25\pi = 78.5 \text{ rad s}^{-1}$$

Find the angular velocity and the end:

$$\omega_2 = 2 \times \pi \times (1500 \text{ min}^{-1} \div 60 \text{ s min}^{-1}) = 50\pi = 157 \text{ rad s}^{-1}$$

Now use the formula:

$$\theta = (\omega_1 + \omega_2)t/2$$

$$\theta = \frac{(78.5 \text{ rad s}^{-1} + 157 \text{ rad}^{-1}) \times 3 \text{ s}}{2} = \mathbf{353 \text{ rad}}$$

The same answer to question 8. I would have been worried if it weren't.

14C.01.11

(a)

Find the angular velocity at the start:

$$\omega_1 = 2 \times \pi \times (60\ \text{min}^{-1} \div 60\ \text{s min}^{-1}) = 2\pi = \mathbf{\underline{6.28\ rad\ s^{-1}}}$$

(b)

Angular acceleration:

$$\omega_2 = \omega_1 + \alpha t$$

$$0 = 6.28\ \text{rad s}^{-1} + 40\ \text{s} \times \alpha$$

$$\alpha = -6.28\ \text{rad s}^{-1} \div 40\ \text{s} = -0.157\ \text{rad s}^{-2}$$

©

The torque is constant, so after 20 s the rate of turning is now 30 min^{-1} = π rad s^{-1}.

Now use the formula:

$$\theta = (\omega_1 + \omega_2)t/2$$

$$\theta = \frac{(6.28\ \text{rad s}^{-1} + 3.14\ \text{rad s}^{-1}) \times 20\ \text{s}}{2} = \mathbf{\underline{94.2\ rad}}$$

(d)

Use $\tau = I\alpha$

$$\tau = 10\,000\ \text{kg m}^2 \times 0.157\ \text{rad s}^{-2} = \mathbf{\underline{1570\ N\ m}}$$

Tutorial 14C.02

14C.02.1

Angular momentum is the product of the moment of inertia (kg m^2) and angular velocity (rad s^{-1}).

Remember that radians are a dimensionless unit, which means they can be ignored.

However, in the exam you will see them as kg m^2 rad s^{-1}.

14C.02.2

Angular momentum = 10 kg m^2 x 20 rad s^{-1} = **200 kg m^2 rad s^{-1}**.

14C.02.3

Since there is zero angular impulse, angular momentum is conserved.

L at start = 1.6 kg m^2 × 5 rad s^{-1} = 8.0 kg m^2 s^{-1}

Moment of inertia of wheel with the clay = 1.6 kg m^2 + 0.25 kg m^2 = 1.85 kg m^2

New angular velocity = 8.0 kg m^2 s^{-1} ÷ 1.85 kg m^2 = **4.32 rad s^{-1}**

14C.02.4

(a)

Since there has been an angular impulse of 1.2 kg m^2 rad s^{-1}, the momentum has increased by 1.2 kg m^2 rad s^{-1}.

Since it was zero to start with, the momentum is now **1.2 kg m^2 rad s^{-1}**.

(b)

$$L = I\omega$$

1.2 kg m^2 rad s^{-1} = 4.8 × 10^{-2} kg m^2 × ω

ω = 1.2 kg m^2 rad s^{-1} ÷ 4.8 x 10^{-2} kg m^2

ω = **25 rad s^{-1}**

©

$\tau = \Delta L/\Delta t$ = 1.2 kg m^2 rad s^{-1} ÷ 2.8 s = **0.43 N m**

14C.02.5

Work done = torque × angle moved = 135 N m × π rad = **424 J**

14C.02.6

Power = torque × angular velocity

Angular Velocity = (3000 min^{-1} ÷ 60 s min^{-1}) × 2π = 100π = 314 rad s^{-1}

Power = 150 Nm × 314 rad s^{-1} = **47 000 W**

14C.02.7

(a)

Energy = power × time = 150 000 W × 4.4 s = **660 000 J**

(b) Use change in kinetic energy:

Energy supplied = difference in kinetic energy

$$E = 1/2\ I\omega_2^2 - 1/2\ I\omega_1^2$$

660 000 J = 1/2 I ((7.4 rad s^{-1})^2 – (1.6 rad s^{-1})^2)

I = 2 × 660 000 J ÷ 52.2 rad^2 s^{-2} = **25 300 kg m^2**

14C.02.8

(a)

(i)

Torque from each jet = 0.60 N × 1.8 m = 1.08 N m

Total torque = 1.08 N m × 4 = **4.32 N m**.

(ii) Torque supplied by the four jets must balance the torque from the frictional couple.

Frictional couple = 4.32 N m

Power = torque × angular velocity.

Angular velocity = 2π rad ÷ 110 s = 0.057 rad s^{-1}

Power = 4.32 N m × 0.057 rad s^{-1} = **0.25 W**

(b)

(i)

Average power from full speed to stop = 0.125 W

Energy = power × time = 0.125 W × 12 s = 1.5 J

(ii)

Energy lost = difference in kinetic energy

$E = 1/2\ I\omega_2^2 - 1/2\ I\omega_1^2$

1.5 J = 1/2 I ($0.057^2 - 0^2$)

I = 2 × 1.5 J ÷ 3.25 × 10^{-3} $rad^2\ s^{-2}$ = **923 kg m^2**

Tutorial 14C.03

14C.03.1

Internal energy is a measure of the vibration of molecules in a material.

In a monatomic gas, it is a measure of kinetic energy of the atoms of gas.

In molecular gases, the internal energy is representative of the sum of kinetic energy of the molecules and the vibration of the bonds.

14C.03.2

Energy supplied = increase in internal energy + work done by system

25 J = ΔU + 20 J

ΔU = **5 J**

14C.03.3

The pump will spring back to where it was.

The gas will cool down, having gained its energy from the surroundings.

14C.03.4

Work out the change in volume:

ΔV= 0.125 m^2 x 0.20 m = 0.025 m^3

Now use $\Delta W = p\Delta V$

ΔW = 1.5 x 10^6 Pa x 0.025 m^3 = **37 500 J**

14C.03.5

The heat flow from the surroundings is negligible compared to the loss of internal energy within the gas.

The loss of gas is very rapid, and the conductivity of the metal is not sufficient to allow a significant heat flow from the surroundings.

Also, the air is a very poor conductor.

14C.03.6

a) The equation to use is:

$$p_1V_1^{\gamma} = p_2V_2^{\gamma}$$

1.01×10^5 Pa × $(4.25 \times 10^{-4}$ m$^3)^{1.40}$ = 1.70×10^5 Pa × $V_2^{1.40}$

1.01×10^5 Pa × 1.90×10^{-5} m^3 = 1.70×10^5 Pa × $V_2^{1.40}$

$$V_2^{1.40} = \frac{1.01 \times 10^5 \text{ Pa} \times 1.90 \times 10^{-5} \text{ m}^3}{1.70 \times 10^5 \text{ Pa}} = 1.13 \times 10^{-5} \text{ m}^3$$

$V_2 = (1.13 \times 10^{-5})^{1/1.4}$ = **2.94×10^{-4} m^3** (which is about 2.9×10^{-4} m^3)

(1/1.4 is the 1.4th root)

(b) Use:

$$\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2}$$

$$\frac{1.01 \times 10^5 \text{ Pa} \times 4.25 \times 10^{-4} \text{ m}^3}{296 \text{ K}} = \frac{1.70 \times 10^5 \text{ Pa} \times 2.94 \times 10^{-4} \text{ m}^3}{T_2}$$

$$T_2 = \frac{296 \times 1.70 \times 10^5 \text{ Pa} \times 2.94 \times 10^{-4} \text{ m}^3}{1.01 \times 10^5 \text{ Pa} \times 4.25 \times 10^{-4} \text{ m}^3}$$

T_2 = **344 K**

14C.03.7

(a)

Work done = pressure × change in volume

Work done = 1.0×10^5 Pa × (150×10^{-4} m^2 × 0.16 m) = **240 J**

(b)

Increase in internal energy = 300 J – 240 J = **60 J**

Did you fall into the bear trap of not converting cm to m and cm^2 to m^2?

Tutorial 14C.04

14C.04.1

(a) Use $pV = nRT$

At A $T_1 = p_1V_1/nR$ = (1.0 × 10^5 Pa × 1 × 10^{-3} m^3)/nR = 100/nR

At B $T_2 = p_2V_2/nR$ = (5.0 × 10^5 × 0.2 × 10^{-3} m^3)/nR = 100/nR

therefore, the temperature are the same, so the line must be an isothermal.

(b)

A to B (heat must be removed to keep temperature the same)

C to A as the pressure falls in a constant volume, and no work is done.

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$W = p\Delta V$ = 5 × 10^5 Pa × (1 × 10^{-3} m^3 – 0.2 × 10^{-3} m^3) = 5 × 10^5 Pa × 0.8 × 10^{-3} m^3 = **400 J**

(d)

(i)

The temperature is the highest at point C. (This is because if *n* remains the same and *R* is a constant, *T* must be high as *P* and *V* are high)

(ii)

$T = pV/nR$ = (5 × 10^5 Pa × 1 × 10^{-3} m^3) ÷ (8.3 J mol^{-1} × 0.069 mol)

T = **870 K**

14C.04.2

Point	*Pressure /Pa*	*Volume /m³*	*Temperature /K*
A	1.0 x 10^5	0.0005	300
B	2.0 x 10^5	0.0005	600
C	2.0 x 10^5	0.0015	1800
D	1.0 x 10^5	0.0015	900

14C.04.3

Area of the rectangle = (0.0015 m^3 – 0.0005 m^{-3}) × (2.0 × 10^5 Pa – 1.0 × 10^5 Pa) = **100 J**

14C.04.4

Thermal efficiency = useful work ÷ heat input

= 100 J ÷ 825 J = 0.121

Percentage efficiency = 0.121 x 100 = **12.1 %**

14C.04.5

Energy per cycle is given by the enclosed area ABCD

Area = 6 × 10^5 Pa × 4.5 × 10^{-3} m^3 = 2700 J

Each cycle takes 0.2 s

Power = 2700 J ÷ 0.2 s = **13 500 W**

Tutorial 14C.05

14C.05.1

(a)

The energy supplied = calorific value × rate of flow

= 45 × 10^6 J kg^{-1} × (2 × 10^{-2} kg min^{-1} ÷ 60 s min^{-1}) = **15 800 J s^{-1}**

(b)

The engine is running at 30 s^{-1}. Engine goes through the power cycle **once every two revolutions**. There are 15 cycles per second:

Indicated power = 380 J × 15 s^{-1} = **5700 J s^{-1}**

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Thermal efficiency = indicated power ÷ input power

= 5700 J s^{-1} ÷ 15 800 J s^{-1} = **0.36** (= 36 %)

14C.05.2

Angular velocity = 2π × (3300 ÷ 60) = 110 π = 346 rad s^{-1}

Power = 250 Nm × 346 rad s^{-1} = 86400 W

Power (in PS = 86400 W ÷ 750 W PS^{-1} = **115 PS**.

14C.05.3

Frictional power = indicated power – power at output.

Frictional power = 5700 W – 4700 W = **1000 W**

This is about **17.5 %** of the indicated power.

14C.05.4

Mechanical efficiency = output power ÷ indicated power

Mechanical efficiency = 4700 W ÷ 5700 W = **0.825** (82.5%).

14C.05.5

Overall efficiency = output power ÷ input power from fuel

Mechanical efficiency = 4700 W ÷ 15800 W = **0.297** (29.7%).

Tutorial 14C.06

14C.06.1

The energy lost to the surroundings = 72 kW – 9 kW = 63 kW.

Efficiency = 9 kW ÷ 72 kW = **0.125** (12.5 %)

14C.06.2

24 °C = 297 K and 100 °C = 373 K

Efficiency = (373 K - 297 K) ÷ 373 K = **0.204** (20.4 %)

14C.06.3

The hot reservoir is the air in the room.

The cold reservoir is the cooling effect of the evaporation of the water on the beak.

(The latent heat of vaporisation of the water as it evaporates cools the beak down and the heat of the room causes the red alcohol to evaporate. The pressure at the bottom causes the fluid to move up the tube and tips the duck forward. When it's forward the fluid runs back down again, making the duck move into an upright position.)

14C.06.4

(a) (i)

Use this equation:

$$\text{Efficiency} = \frac{T_H - T_C}{T_H}$$

Efficiency = (360 K - 280 K) ÷ 360 K = **0.222** (= 22.2 %)

(ii)

To get 5 MW, rate of energy exchange must be:

Heat flow = 5.0 ÷ 0.222 = **22.5 MW**

(iii)

Rate at which energy is passed to seawater = 22.5 MW - 5.0 MW = **17.5 MW**

(b) Any two of

There will be friction within the heat engine.

There will be heating in the generator windings as a current passes through the wires.

Losses to the atmosphere.

Variations in sea temperature.

(c) Any one of

Price of oil is expensive.

It has to be transported to the site.

Waste products might have to be treated.

14C.06.5

(a) The definition is:

the ratio of the heat transferred to the **hot reservoir** to the input energy required to pump the heat from the cold reservoir to the hot reservoir.

(You will lose a mark for saying "heat input" as it's too vague.)

(b) (i) Work out the maximum theoretical efficiency. Equation to use:

$$\text{Efficiency} = \frac{T_H - T_C}{T_H}$$

Efficiency = (1600 K - 290 K) ÷ 1600 K = **0.81875** (= **82 %**)

Now work out the input energy (per second):

E_{in} = 80 000 W ÷ 0.81875 = **97710 W** (= 98 kW)

Now we can work out the fuel flow:

Fuel flow per second = energy per second ÷ calorific value

= 97710 W ÷ 49 ×10^6 K kg^{-1} = 1.994 × 10^{-3} kg s^{-1} (= **2.0 × 10^{-3} kg s^{-1}**)

(i.e. 2 grams per second).

(ii)

In this system, the waste heat from the engine is used to heat the hot space, as well as the output from the heat pump.

Heat flow from cold to hot = 16000 W × 2.6 = 41600 W (= 42 kW)

Heat loss from the engine = 98 kW - 80 kW = 18 kW.

Total energy flow = 42 kW + 18 kW = **60 kW**

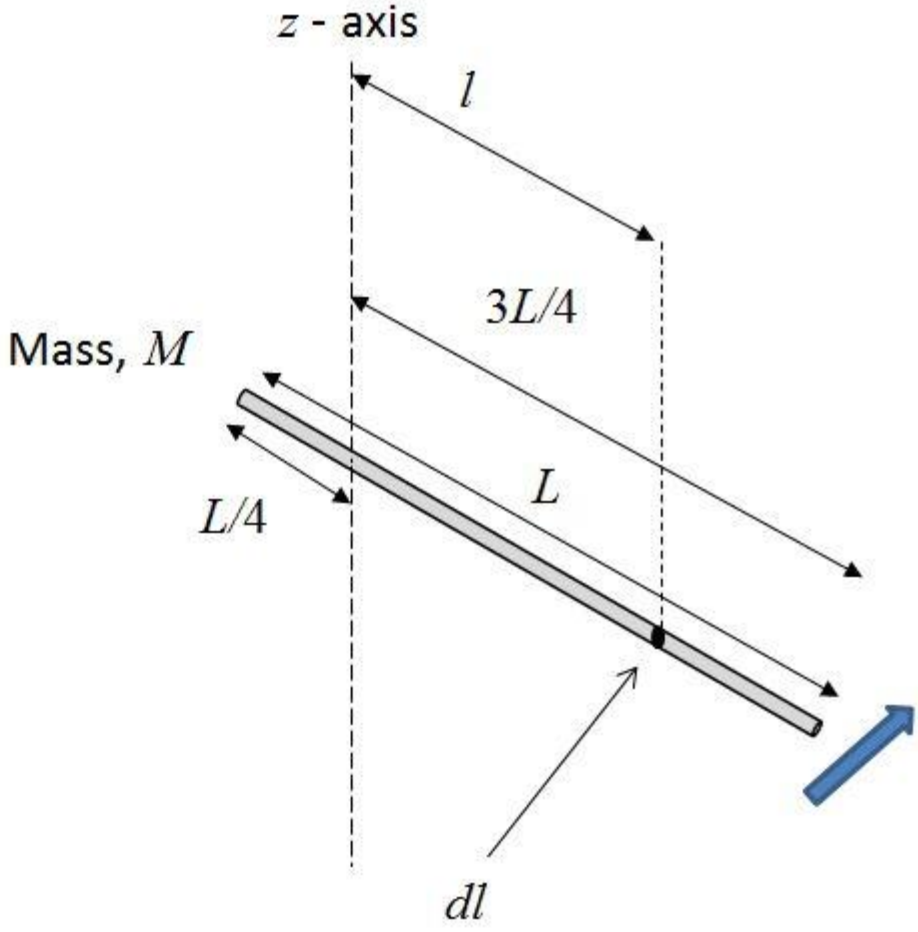
(iii)

The amount of electrical energy available to heat the hot space would be only 16 kW compared with 60 kW with this system. The system adds energy from external sources.

Tutorial 14C.07

14C.07.1

(a) Draw a diagram to show the system:



We know the definition of the moment of inertia is given by:

$$I_T = \int_{r_1}^{r_2} r^2 dm$$

We will consider very small element of length dl that is l from the z-axis. The element has a very small mass dm. Since the rod is uniform, we can say that:

$$\mu = \frac{M}{L} = \frac{dm}{dl}$$

We can rearrange this to give:

$$dm = \frac{M}{L} dl$$

(Yes, I did do a copy and paste from the text.)

We can modify our general equation by writing

- $r = l$
- $r_1 = -L/4$
- $r_2 = +3L/4$

The terms M and L are constant. So, the equation becomes:

$$I = \frac{M}{L} \int_{-L/4}^{+3L/4} l^2 dl$$

This integrates to:

$$I = \frac{M}{L}\left[\left(\frac{\left(+3L/4\right)^3}{3}\right) - \left(\frac{\left(-L/4\right)^3}{3}\right)\right]$$

This simplifies to:

$$I = \frac{M}{3L}\left[\left(\frac{+3L}{4}\right)^3 - \left(\frac{-L}{4}\right)^3\right]$$

Therefore:

$$I = \frac{M}{3L}\left(\frac{27L^3}{64} + \frac{L^3}{64}\right)$$

Cancelling the *L* terms and tidying gives:

$$I = \frac{28ML^2}{192}$$

The common factor between 28 and 192 is 4, so we divide top and bottom by 4:

$$I = \frac{7ML^2}{48}$$

(b)

Area = $\pi \times (0.5 \times 10^{-2}\ \text{m})^2 = 7.85 \times 10^{-5}\ \text{m}^2$

Volume = $7.85 \times 10^{-5}\ \text{m}^2 \times 1.2\ \text{m} = 9.42 \times 10^{-5}\ \text{m}^3$

Mass = volume × density = $9.42 \times 10^{-5}\ \text{m}^3 \times 7600\ \text{kg m}^{-3}$ = **0.716 kg**

(c)

$I = (7 \times 0.716\ \text{kg} \times (1.2\ \text{m})^2) \div 48 = 0.150\ \text{kg m}^2$ = **$0.15\ \text{kg m}^2$** (to 2 s.f.)